Bode-Like Integral for Stochastic Switched Systems in the Presence of Limited Information

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Abstract

In this paper, we establish a Bode sensitivity integral formula for a class of feedback closed-loop systems with stochastic switched plants and controllers. Using information theory, we derive an information conservation law, based on which a log integral theorem is obtained for the closed loops of interest. Furthermore, we develop several algebraic conditions to explicitly capture the performance limitations. Networked Control Systems (NCS) are used as an illustrative example.

I. Introduction

Recent results on fundamental limitations of feedback control in the presence communication channels presented a fairly general and complete approach, in discrete-time setting, towards unification of information theory and control theory, [1], [2]. Entropy rate inequalities corresponding to the information flux in a typical causal closed-loop configuration were derived towards obtaining a Bode-like integral formula. Prior to this, extensions of Bode's theorem have been claimed for certain discrete-time nonlinear systems and linear time-varying systems respectively, [3], [4].

In this paper, we extend the framework from [2] to closed loops with stochastic switched plants. While switched control systems have been studied from various perspectives [5], it is still not clear how to characterize their fundamental limitations within an appropriate framework similar to the Bode's formula. This work is the first attempt, to the best of the authors' knowledge, to address this yet to be solved problem. Throughout the paper, we restrict the switching pattern that governs the transitions among (finite) subsystems to a finite state homogenous Markov chain, which is further assumed to admit an invariant measure that characterizes its long-term average behavior. This class of stochastic switched systems (or variations with similar probabilistic switching laws) is important in itself, and has been attracting never-fading attention since its inception for its theoretical richness and deep rooting in practice. The fundamental control problems such as stochastic stability [6], [7], optimal control [8] and filtering [9] have been extensively studied in both continuous-time and discrete-time settings. On the more practical side, among other applications ranging from regime changing models in macro economics [10] to single-event aversion in hight altitude aerospace systems [11], its importance is recently re-discovered for providing an appropriate model for networked control systems (NCS) with random data-outage caused by either packet drop-outs or communication delays [12]–[14].

The development of the paper begins with some necessary technical assumptions that allow the system of interest being studied by information theory. The machineries from information theory are intensively employed towards deriving a conservation law, in the form of an entropy rate inequality, that fundamentally governs the disturbance attenuation ability of the closed loop regardless the choice of the controller. To obtain the information conservation law, arguments similar to [15] and [2] are adopted to explore the probabilistic dependencies among signals residing in the closed loop. Those sequential relations, as will be revealed in the main result, are accounted by the closed loop topology and causality. The novelty of this paper as compared with predecessors is in the development of a new approach for handling the Markovian switching sequence. From the information theoretical point of view, the underlying Markovian switching sequence can be regarded as an additional information source [16]. Not surprisingly, the resulting information conservation law is expected to show the deviation from prior results brought up by this extra information influx. While the conservation law we obtain is generic enough to include any nonlinear closed loops with a given structure, it is of interest to show that it recovers Bode's integral for some special cases. More specifically, we impose the (wide-sense) stationarity condition on the relevant signals to allow for the existence of their *power spectral densities*, the ratio among which can be conveniently deemed as analogies of transfer functions in the classical sense. Together with these additional conditions, the conservation law immediately yields the needed logarithmic integral, completing one side of Bode's formula. For the other side of Bode's inequality (or equality in some cases), one needs to evaluate a mutual information rate between input error signal (between control signal and its disturbed version) and inherent information source (initial condition and switching sequence) of the plant. A lower bound of this mutual information rate is then obtained as a Lyapuonov-exponent-like quantity, which can be understood as a generalized *degree of instability*. Notice that the degree of instability has been widely accepted as the sum of the logarithmic magnitudes of unstable eigenvalues in the LTI setting [17]. However, as pointed out in [18], [19], it is difficult to calculate the Lyapunov exponents of time-varying or switched systems in general. Therefore, in order to obtain a closed-form expression of the Lyapunov exponent (or its lower bound), additional conditions are inevitable. In this case we exploit the properties of the Lie algebra generated by system matrices, which are widely studied over the past decade with abundant results for both linear and nonlinear systems [5], [20]. Both sides of Bode's formula are then completed as an explicit expression of the Lyapunov exponent is obtained. To demonstrate the usefulness of the theoretical result, we propose an example of NCS with packet dropouts, for which various estimation and compensation schemes have been developed to cope with the random data loss [12], [21]. The application of our Bode's formula shows that the degree of instability of the plants determines the lower bound of the performance limitation no matter what forms of stable compensator are chosen.

The paper is organized as follows. In Section II we introduce the closed-loop feedback configuration and some basic definitions and facts, Section III provides the problem formulation, Section IV studies a general feedback scheme, within which we develop a mutual information inequality and a Bode-type integral formula. Section V applies Bode's integral to NCS. The paper is concluded in Section VI.

Notation:

- $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ stands for complex plane; $\mathbb{C}^-$ and $\mathbb{C}^+$ stand for the left half and right half of $\mathbb{C}$ respectively.
- Random variables defined in appropriate probability spaces are represented using boldface letters, such as $\mathbf{x}$, $\mathbf{y}$. If not otherwise stated, the random variables take values in $\mathbb{R}$ throughout the paper.
- If $\mathbf{x}(k), k \in \mathbb{N}^+$, is a discrete time stochastic process, we denote its segment $\{\mathbf{x}(k)\}_{k=l}^{u}$ by $\mathbf{x}_l^u$, and use $\mathbf{x}_0^n := \mathbf{x}^n$ for simplicity.
- $\mathbf{E}[\cdot]$ is the expectation operator of a random variable.
- $(\cdot)^+ = \max\{\cdot, 0\}$ and $(\cdot)^- = \min\{\cdot, 0\}$.
- $\Re(\cdot)$ gives the real part of a complex number.
- $\lambda_j(\cdot)$ gives the eigenvalues of a square matrix.
- $h(\cdot)$ stands for (differential) entropy and $I(\cdot;\cdot|\cdot)$ for conditioned mutual information; $\bar{h}$ and $\bar{I}$ stand for the entropy rate and mutual information rate respectively.
- When A is a finite set, $|A|$ gives the number of elements in A.
- $sp\{\cdot\}$ denotes the spectrum of an operator.

II. Preliminaries

We begin by introducing some elementary definitions and results from information theory, most of which are taken from [22].

Definition 2.1 (Differential Entropy): The *differential entropy* of a continuous random variable $\mathbf{x}$ with density $p_{\mathbf{x}}$ is defined as

$$h(\mathbf{x}) := -\mathbf{E}[\log p_{\mathbf{x}}] = -\int_{\mathbb{S}} p_{\mathbf{x}} \log p_{\mathbf{x}} d\mathbf{x}\,,$$

where $\mathbb{S}$ is an abstract space where the random variable $\mathbf{x}$ is defined.

Definition 2.2 (Conditional Entropy): If there are two random variables $\mathbf{x}$ and $\mathbf{y}$, the conditional entropy $h(\mathbf{x}|\mathbf{y})$ is defined as

$$h(\mathbf{x}|\mathbf{y}) := -\int_{\mathbb{S}^2} p_{\mathbf{xy}} \log p_{\mathbf{x}|\mathbf{y}} d\mathbf{x} d\mathbf{y}$$

Definition 2.3 (Joint Entropy): The entropy of the random vector $\mathbf{x}^n := \mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n$, comprised of random variables with density $p_{\mathbf{x}^n}$, is defined as

$$h(\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n) := -\mathbf{E}[\log p_{\mathbf{x}^n}] = -\int_{\mathbb{S}^n} p_{\mathbf{x}^n} \log p_{\mathbf{x}^n} d\mathbf{x}^n$$

Definition 2.4 (Mutual Information): The mutual information between the two random variables $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$I(\mathbf{x};\mathbf{y}) := -\mathbf{E}_{\mathbf{xy}}\left[\log \frac{p_{\mathbf{xy}}}{p_{\mathbf{x}}p_{\mathbf{y}}}\right] = -\int_{\mathbb{S}^2} p_{\mathbf{xy}} \log \frac{p_{\mathbf{xy}}}{p_{\mathbf{x}}p_{\mathbf{y}}} d\mathbf{x} d\mathbf{y}$$

Definition 2.5 (Conditional Mutual Information): The mutual information between the two random variables $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\begin{aligned} I(\mathbf{x};\mathbf{y}|\mathbf{z}) &:= -\mathbf{E}_{\mathbf{xyz}}\left[\log \frac{p_{\mathbf{xy}|\mathbf{z}}}{p_{\mathbf{x}|\mathbf{z}}p_{\mathbf{y}|\mathbf{z}}}\right] \\ &= -\int_{\mathbb{S}^3} p_{\mathbf{xyz}} \log \frac{p_{\mathbf{xy}|\mathbf{z}}}{p_{\mathbf{x}|\mathbf{z}}p_{\mathbf{y}|\mathbf{z}}} d\mathbf{x} d\mathbf{y} d\mathbf{z} \end{aligned}$$

Definition 2.6 (Joint Mutual Information): The joint mutual information between n dimensional vectors $\mathbf{x}^n := \mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_n$ and $\mathbf{y}^n := \mathbf{y}_0, \mathbf{y}_1, ..., \mathbf{y}_n$ is defined as

$$\begin{aligned} I(\mathbf{x}^n;\mathbf{y}^n) &= -\mathbf{E}_{\mathbf{x}^n\mathbf{y}^n}\left[\log \frac{p_{\mathbf{x}^n\mathbf{y}^n}}{p_{\mathbf{x}^n}p_{\mathbf{y}^n}}\right] \\ &= -\int_{\mathbb{S}^{2n}} p_{\mathbf{x}^n\mathbf{y}^n} \log \frac{p_{\mathbf{x}^n\mathbf{y}^n}}{p_{\mathbf{x}^n}p_{\mathbf{y}^n}} d\mathbf{x}^n d\mathbf{y}^n \end{aligned}$$

Definition 2.7: [Entropy Rate] The entropy rate of $\mathbf{x}$ is defined as

$$\bar{h}(\mathbf{x}) := \lim_{n\to\infty} \frac{h(\mathbf{x}^n)}{n+1}\,,$$

given the existence of the limit.

Definition 2.8 (Mutual Information Rate): The mutual information rate of two stochastic processes is defined as

$$\bar{I}(\mathbf{x};\mathbf{y}) := \lim_{n\to\infty} \frac{I(\mathbf{x}^n;\mathbf{y}^n)}{n+1}\,,$$

given the existence of the limit.

We also introduce some widely used properties [22]:

(P1) *Symmetry and nonnegativity*:
$I(\mathbf{x};\mathbf{y}) = I(\mathbf{y};\mathbf{x}) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) \geq 0\,.$

(P2) *Kolmogorov equality*:

$$I(\mathbf{x};(\mathbf{y},\mathbf{z})) = I(\mathbf{x};\mathbf{z}) + I(\mathbf{x};\mathbf{y}|\mathbf{z})$$

(P3) *Data processing inequality*:

$$I(\mathbf{x};\mathbf{y}|\mathbf{z}) \geq I(\mathbf{x};g(\mathbf{y})|\mathbf{z})$$

The equality holds, if $g(\cdot)$ is invertible.

(P4) *Invariance of mutual information (entropy)*

$$I(\mathbf{x};\mathbf{y}|\mathbf{z}) = I(\mathbf{x}+g(\mathbf{z});\mathbf{y}|\mathbf{z})\,, h(\mathbf{x}|\mathbf{z}) = h(\mathbf{x}+g(\mathbf{z})|\mathbf{z}),$$

where $g(\cdot)$ is a function.

(P5) *Chain rule*:

$$h(\mathbf{x}^n|\mathbf{y}) = \sum_{k=1}^{n} h(\mathbf{x}_k|\mathbf{y},\mathbf{x}^{k-1})$$

(P6) *Maximum entropy*: Consider $\mathbf{x} \in \mathbb{R}^m$ and the covariance matrix given by $V := \mathbf{E}[\mathbf{x}\mathbf{x}^\top]$. Then we have

$$h(\mathbf{x}) \leq h(\bar{\mathbf{x}}) = \frac{1}{2}\log((2\pi e)^m \det V)\,,$$

where $\bar{\mathbf{x}}$ is a Gaussian process with the same covariance as $\mathbf{x}$. Equality holds, if $\mathbf{x}$ is Gaussian.

Definition 2.9 (Wide Sense Stationary Process): A zero-mean stochastic process $\mathbf{x}(k) \in \mathbb{R}^n$, $k \geq 0$ is stationary, if for all $k \geq 0$ its covariance function, defined by

$$R_{\mathbf{x}}(l) = \mathbf{E}[\mathbf{x}(k+l)\mathbf{x}^\top(k)], \quad l \in \mathbb{N}^+\,,$$

is independent of k. Throughout this paper, *wide sense stationary* is abbreviated as *stationary* for convenience.

Definition 2.10: The spectral density of a stationary process $\mathbf{v}$ is given as the following Fourier transform

$$f_{\mathbf{v}}(\omega) = \frac{1}{2\pi}\sum_{k=0}^{\infty}\mathbf{v}(k)e^{-j\omega k}$$

Definition 2.11 (Sensitivity-like Function): A sensitivity-like function of the closed loop is defined as

$$S_{\mathbf{d},\mathbf{e}}(\omega) = \sqrt{\frac{f_{\mathbf{e}}(\omega)}{f_{\mathbf{d}}(\omega)}}\,,$$

where $\mathbf{e}$ and $\mathbf{d}$ are stationary and stationarily correlated.

Remark 2.12: The function $S_{\mathbf{d},\mathbf{e}}(\omega)$ is the stochastic analogue of the sensitivity function $|S(j\omega)|$ in Bode's original work [23].

Throughout, we adopt the following stability definition.

Definition 2.13 (Mean-square Stability): The closed loop given in Fig. 1 is said to be mean-square stable, if

$$\sup_{k\geq 0}\mathbf{E}[\mathbf{x}^\top(k)\mathbf{x}(k)] < \infty\,. \tag{1}$$

Definition 2.14 (Lie Algebra): A Lie algebra is denoted as

$$\mathfrak{g} := \{A(n) : n \in \mathcal{N}\}_{LA}\,,$$

which is generated by the matrices $A(n), n \in \mathcal{N}$, with respect to the standard Lie bracket

$$[A(1), A(2)] := A(1)A(2) - A(2)A(1)\,.$$

We say that the Lie algebra $\mathfrak{g}$ is *solvable* if the following derived series

$$\mathfrak{g} > [\mathfrak{g},\mathfrak{g}] > [[\mathfrak{g},\mathfrak{g}],[\mathfrak{g},\mathfrak{g}]] > \ldots$$

becomes 0 in finite operations, where $[\cdot,\cdot]$ denotes the algebra generated by Lie bracket and "$>$" denotes the partial order of sub-algebra

Theorem 2.15 (Simultaneous triangularization [5]): The matrices $\{A(n) : n \in \mathcal{N}\}$ can be simultaneously triangularized by some linear operator $T \in \mathbb{C}^{m\times m}$, if and only if the Lie algebra $\mathfrak{g}$ is *solvable*.

III. Problem Formulation

Throughout the paper we consider the feedback configuration depicted in Fig. 1. With the following

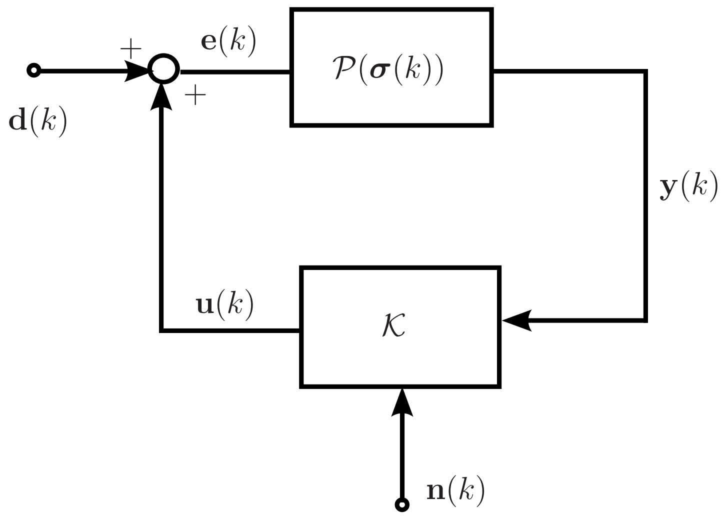

Fig. 1. Basic Feedback Scheme

assumptions

- The plant $\mathcal{P}(\boldsymbol{\sigma}(k))$ is modeled by the following stochastic difference equation

$$\begin{aligned} \mathbf{x}(k+1) &= A(\boldsymbol{\sigma}(k))\mathbf{x}(k) + B(\boldsymbol{\sigma}(k))\mathbf{e}(k)\,, \quad \mathbf{x}(0) = \mathbf{x}_0\,, \\ \mathbf{y}(k) &= C(\boldsymbol{\sigma}(k))\mathbf{x}(k)\,, \quad k = 0,1,2.... \end{aligned} \tag{2}$$

  Here $\mathbf{x}(k) \in \mathbb{R}^m$, and $\mathbf{x}_0$ is assumed to have finite differential entropy or $h(\mathbf{x}_0) < \infty$, and $\boldsymbol{\sigma}(k) \in \{1,2,...,N\} =: \mathcal{N}$ is a finite state ergodic Markov process given by

$$P(\boldsymbol{\sigma}(k+1) = j|\sigma(k) = i) := p_{ij} \geq 0\,,$$

  where p_{ij} is named as transition probability from state i to j, and $\sum_j p_{ij} = 1$ for all $i \in \mathcal{N}$. The stationary distribution of the Markov chain $\boldsymbol{\sigma}$, denoted as $\boldsymbol{\pi} = [\pi_1, ..., \pi_{|\mathcal{N}|}]$, is obtained by solving

$$\boldsymbol{\pi}^\top [p_{ij}]_{i,j\in\mathcal{N}} = \boldsymbol{\pi}^\top\,, \text{ and } \quad [1,..,1]\boldsymbol{\pi} = 1\,.$$

  We also assume $A(n), n \in \mathcal{N}$, is not singular.
- The disturbance $\mathbf{d}(k)$ is a stochastic process, and $\mathbf{n}(k)$ is a stochastic process that models the observation noise. We assume that $\boldsymbol{\sigma}(k)$, $\mathbf{d}(k)$, $\mathbf{n}(k)$ and $\mathbf{x}_0$ are mutually independent.
- The controller $\mathcal{K}$ is given as a deterministic causal map such that

$$\mathcal{K} : (k, \mathbf{y}^{k-1}, \mathbf{n}^k) \mapsto \mathbf{u}(k)\,.$$

This paper aims at developing a Bode-like formula for the above described stochastic switched system to capture its disturbance attenuation property, which is fundamentally determined by the closed-loop structure and the open-loop plant dynamics.

IV. Main Results

In this section we develop the information conservation law of the closed-loop system depicted in Fig. 1. In turn, an analogue of Bode's formula is obtained under the assumption of stationarity.

A. Information Conserved Closed Loop

The following lemma is introduced to characterize the closed-loop causality.

Lemma 4.1:

$$I(\mathbf{d}(i);(\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i)|\mathbf{d}^{i-1})=0\,,\quad \forall\ i\geq 1. \tag{3}$$

Proof:

$$\begin{aligned}
&I(\mathbf{d}(i);(\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i)|\mathbf{d}^{i-1})\\
&\overset{(a)}{\leq} I(\mathbf{d}(i);(\mathbf{u}^i,\mathbf{n}^i,\boldsymbol{\sigma}^i,\mathbf{x}_0)|\mathbf{d}^{i-1})\\
&\overset{(b)}{\leq} I(\mathbf{d}(i);(\mathbf{d}^{i-1},\mathbf{n}^i,\boldsymbol{\sigma}^i,\mathbf{x}_0)|\mathbf{d}^{i-1})\\
&\overset{(c)}{=} I(\mathbf{d}(i);(\mathbf{n}^i,\boldsymbol{\sigma}^i,\mathbf{x}_0)|\mathbf{d}^{i-1})\\
&\overset{(d)}{=} 0
\end{aligned}$$

Here, (a) follows from (P3); (b) also follows from (P3), since $\mathbf{u}^i$ is a function of $(\mathbf{d}^{i-1},\mathbf{n}^i,\boldsymbol{\sigma}^i,\mathbf{x}_0)$; (c) follows from (P4), and (d) is implied because $\mathbf{n}$, $\boldsymbol{\sigma}$, $\mathbf{x}_0$ and $\mathbf{d}$ are mutually independent. ■

In what follows we use the result from Lemma 4.1 to achieve an equality, revealing a key relationship among the signals residing in Fig. 1.

Lemma 4.2: Consider the closed loop in Fig. 1. The following inequality holds

$$h(\mathbf{e}^k)=h(\mathbf{d}^k)+I((\mathbf{x}_0,\boldsymbol{\sigma}^k);\mathbf{e}^k)+\sum_{i=1}^{k} I(\mathbf{u}^i;\mathbf{e}(i)|\mathbf{e}^{i-1},\mathbf{x}_0,\boldsymbol{\sigma}^k) \tag{4}$$

Proof: We break down the equality (3) by

$$\begin{aligned}
0&=I(\mathbf{d}(i);(\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i)|\mathbf{d}^{i-1})\\
&\overset{(a)}{=} I(\mathbf{d}(i);\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i,\mathbf{d}^{i-1})-I(\mathbf{d}(i);\mathbf{d}^{i-1})\\
&\overset{(b)}{=} I(\mathbf{d}(i);\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i,\mathbf{e}^{i-1})-I(\mathbf{d}(i);\mathbf{d}^{i-1})\\
&\overset{(c)}{=} -h(\mathbf{d}(i)|\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i,\mathbf{e}^{i-1})+h(\mathbf{d}(i)|\mathbf{d}^{i-1})\\
&\overset{(d)}{=} -h(\mathbf{e}(i)|\mathbf{u}^i,\mathbf{x}_0,\boldsymbol{\sigma}^i,\mathbf{e}^{i-1})+h(\mathbf{d}(i)|\mathbf{d}^{i-1})\\
&\overset{(e)}{=} -h(\mathbf{e}(i)|\mathbf{e}^{i-1})+I((\mathbf{x}_0,\boldsymbol{\sigma}^i));\mathbf{e}(i)|\mathbf{e}^{i-1})+\\
&\qquad I(\mathbf{u}^i;\mathbf{e}(i)|\mathbf{e}^{i-1},\mathbf{x}_0,\boldsymbol{\sigma}^i)+h(\mathbf{d}(i)|\mathbf{d}^{i-1})\,.
\end{aligned}$$

Here (a) follows from (P3), (b) follows from the fact that $\mathbf{e}^{i-1}=\mathbf{u}^{i-1}+\mathbf{d}^{i-1}$, (c) follows from (P1), (d) follows from (P4) and (e) from (P5). Summing up the above equality from 1 to k and using (P5), we have (4). ■

Remark 4.3: The term $\sum_{i=1}^{k} I(\mathbf{u}^i;\mathbf{e}(i)|\mathbf{e}^{i-1},\mathbf{x}_0,\boldsymbol{\sigma}^k)$ is alternatively represented as the directed information from $\mathbf{u}$ to $\mathbf{e}$ conditioned by $(\mathbf{x}_0,\boldsymbol{\sigma}^k)$ [24].

Theorem 4.4: Consider the closed loop shown in Fig. 1. The following entropy rate inequality holds

$$\bar{h}(\mathbf{e})\geq \bar{h}(\mathbf{d})+\bar{I}((\mathbf{x}_0,\boldsymbol{\sigma});\mathbf{e})\,. \tag{5}$$

Proof: Considering the nonegativeness of the mutual information, from (4) we have

$$h(\mathbf{e}^k) \geq h(\mathbf{d}^k) + I((\mathbf{x}_0, \boldsymbol{\sigma}^k); \mathbf{e}^k)\,.$$

The proof is completed by taking the limit of

$$\frac{h(\mathbf{e}^k)}{k+1} \geq \frac{h(\mathbf{d}^k)}{k+1} + \frac{I((\mathbf{x}_0, \boldsymbol{\sigma}^k); \mathbf{e}^k)}{k+1}$$

as $k \to \infty$. ■

Remark 4.5: The inequality in (5) has been derived in both information theory and control theory literature in different setups and with different generalities [2], [15], [25]. In [25], a similar Lyapunov exponent is used to deal with time-varying systems.

B. Degree of Instability

As it can be seen in (5), the mutual information rate $\bar{I}((\mathbf{x}_0, \boldsymbol{\sigma}); \mathbf{e})$ plays an important role in the conservation law. In this subsection we establish some nontrivial lower bounds of this mutual information rate, and possible closed-form expressions are obtained with additional algebraic conditions.

To begin with, we have the following lower bound.

Theorem 4.6: Consider the closed loop in Fig. 1. The following inequality holds.

$$\bar{I}((\mathbf{x}_0, \boldsymbol{\sigma}); \mathbf{e}) \geq \liminf_{k\to\infty} \frac{1}{k+1} \mathbf{E} \sum_j \Re \left(\log \lambda_j \left(F_k\right)\right)^+ \,, \tag{6}$$

where $F_k := A(\boldsymbol{\sigma}(k))A(\boldsymbol{\sigma}(k-1)) \cdots A(\boldsymbol{\sigma}(0))$.

Proof: We first consider the dynamics of the plant

$$\mathbf{x}(k+1) = \mathbf{x}(k)A(\boldsymbol{\sigma}(k)) + B(\boldsymbol{\sigma}(k))\mathbf{e}(k)\,,$$

which can be solved as

$$\begin{aligned}\mathbf{x}(k+1) = &\left(\prod_{i=0}^{k} A(\boldsymbol{\sigma}(i))\right)\mathbf{x}_0 + \\ &\sum_{i=0}^{k}\left(\prod_{l=i}^{k} A(\boldsymbol{\sigma}(l))\right) B(\boldsymbol{\sigma}(i))\mathbf{e}(i) \\ &= F_k(\mathbf{x}_0 - \hat{\mathbf{x}}_0(k+1)),\end{aligned}$$

where

$$\begin{aligned}&\hat{\mathbf{x}}_0(k+1) := \\ -&\left(\prod_{i=0}^{k} A(\boldsymbol{\sigma}(i))\right)^{-1} \sum_{i=0}^{k}\left(\prod_{l=i}^{k} A(\boldsymbol{\sigma}(l))\right) B(\boldsymbol{\sigma}(i))\mathbf{e}(i)\,.\end{aligned}$$

F_k can be decomposed into the following form by a linear transformation T_k:

$$T_k^{-1} F_k T_k = \begin{bmatrix} F_{ku} & 0 \\ 0 & F_{ks} \end{bmatrix},$$

where F_{ku} is unstable and F_{ks} is stable. The same linear transformation can be applied to $\mathbf{x}_0$ and $\hat{\mathbf{x}}_0$ to obtain

$$T_k\mathbf{x}_0 = \begin{bmatrix} \mathbf{x}_{u0} \\ \mathbf{x}_{s0} \end{bmatrix} \qquad \text{and } T_k\hat{\mathbf{x}}_0 = \begin{bmatrix} \hat{\mathbf{x}}_{u0} \\ \hat{\mathbf{x}}_{s0} \end{bmatrix}.$$

We note that $\mathbf{x}_{u0}$ and $\mathbf{x}_{s0}$ are functions of k, however this argument is omitted for notational simplicity.

We then establish the lower bound of $I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k)$ as follows

$$\begin{aligned}
&I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) \\
&\overset{(a)}{=} I(\mathbf{x}_0; \boldsymbol{\sigma}^k) + I(\mathbf{x}_0; \mathbf{e}^k|\boldsymbol{\sigma}^k) \\
&\overset{(b)}{=} I(\mathbf{x}_0; \mathbf{e}^k|\boldsymbol{\sigma}^k) \\
&\overset{(c)}{=} I(\mathbf{x}_{u0}, \mathbf{x}_{s0}; \mathbf{e}^k|\boldsymbol{\sigma}^k) \\
&\overset{(d)}{=} h(\mathbf{x}_{u0}, \mathbf{x}_{s0}|\boldsymbol{\sigma}^k) - h(\mathbf{x}_{s0}|\mathbf{x}_{u0}, \mathbf{e}^k, \boldsymbol{\sigma}^k) - h(\mathbf{x}_{u0}|\mathbf{e}^k, \boldsymbol{\sigma}^k)
\end{aligned}$$

where (a) follows from (P2), (b) follows from the fact that $\mathbf{x}_0$, $\mathbf{e}$ and $\boldsymbol{\sigma}$ are mutually independent, (c) follows from the invariance of mutual information under linear transformation, (d) follows from (P2) and the independence between $\mathbf{x}_0$ and $\boldsymbol{\sigma}$.

We further evaluate the term $h(\mathbf{x}_{u0}|\mathbf{e}^k, \boldsymbol{\sigma}^k)$ by

$$\begin{aligned}
&h(\mathbf{x}_{u0}|\mathbf{e}^k, \boldsymbol{\sigma}^k) \\
&\overset{(a)}{=} h(\mathbf{x}_{u0} - \hat{\mathbf{x}}_{u0}|\mathbf{e}^k, \boldsymbol{\sigma}^k) \\
&\overset{(b)}{\le} h(\mathbf{x}_{u0} - \hat{\mathbf{x}}_{u0}) \\
&\overset{(c)}{\le} \log(2\pi e)^{l_k} - \log \mathbf{E}|\det F_{ku}| + \log \mathbf{E} \det \mathbf{x}_{u0}(k)\mathbf{x}_{u0}^{\top}(k) \\
&\overset{(d)}{\le} \log(2\pi e)^{l_k} - \mathbf{E} \log|\det F_{ku}| + \log \mathbf{E} \det \mathbf{x}_{u0}(k)\mathbf{x}_{u0}^{\top}(k),
\end{aligned}$$

where $0 \le l_k \le m$ is the dimension of $\mathbf{x}_{u0}$, $\mathbf{x}_{u0}(k)$ is the vector formed by the first l_k elements of $T_k\mathbf{x}(k)$ and the last inequality follows form Jensen's inequality. Similar argument shows that $h(\mathbf{x}_{s0}|\mathbf{x}_{u0}, \mathbf{e}^k, \boldsymbol{\sigma}^k) < \infty$ for all k. We also notice that $h(\mathbf{x}_{u0}, \mathbf{x}_{s0}|\boldsymbol{\sigma}^k) < h(\mathbf{x}_0) < \infty$.

In what follows we have

$$\begin{aligned}
I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) \ge & -\log(2\pi e)^{l_k} + \mathbf{E} \log|\det F_{ku}| \\
& - \log \mathbf{E} \det \mathbf{x}_{u0}(k)\mathbf{x}_{u0}^{\top}(k) - h(\mathbf{x}_{s0}|\mathbf{x}_{u0}, \mathbf{e}^k, \boldsymbol{\sigma}^k)\,.
\end{aligned}$$

Note that the stability of the closed-loop system implies that $\mathbf{E} \det \mathbf{x}_{u0}(k)\mathbf{x}_{u0}^{\top}(k) < \infty\,, \forall k$. Then we have

$$\begin{aligned}
\bar{I}(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) &\ge \liminf_{k\to\infty} \frac{1}{k+1} \mathbf{E} \log|\det F_{ku}| \\
&= \liminf_{k\to\infty} \frac{1}{k+1} \mathbf{E} \sum_j \Re \left(\log \lambda_j \left(F_k\right)\right)^+ \,.
\end{aligned}$$

■

Remark 4.7: The right hand side of (6) is actually a Lyapunov exponent for the dynamical system in (2). For a complete treatment of Lyapunov exponents for stochastic switching systems, the reader can refer to [19]. The similar expression has also been obtained as infimum rate for controlling a Markov jump linear system with finite data-rate [26].

Remark 4.8: Here we stress that the Markov chain $\boldsymbol{\sigma}$ can be viewed as a source of uncertainty (or randomness) that contributes to the overall performance limitation of the closed loop. The information source of this probabilistic dynamics has also been studied in the area of communication theory, and some recent development shows that by using Lyapunov exponent of a product random matrices [27], one can effectively calculate the related entropy rate and mutual information rate, which ultimately lead to the *channel capacity*. However, the Lyapunov exponent in our work here is used to capture the open-loop instability induced by the dynamics of the plants on individual operating modes together with the switching sequence.

To overcome the challenge of obtaining $\liminf_{k\to\infty}\frac{1}{k+1}\mathbf{E}\sum_j \Re\left(\log\lambda_j\left(F_k\right)\right)^+$ by using the method of Lyapunov exponents, we exploit the algebraic structure of matrices $A(n), n\in\mathcal{N}$. From Theorem 2.15 we know that the solvability of $\mathfrak{g}$ implies that $\{A(n)\}, n\in\mathcal{N}$, can be simultaneously triangularizable by some linear transformation $T\in\mathbb{C}^{m\times m}$:

$$T^{-1}A(n)T=\begin{bmatrix}\lambda_1^{(n)} & \star & \star\\ 0 & \ddots & \star\\ 0 & 0 & \lambda_m^{(n)}\end{bmatrix},\forall n\in\mathcal{N}\,. \tag{7}$$

Next we divide the index set $\{1,...,m\}$ into two distinct sets $\mathcal{M}_u$ and $\mathcal{M}_s$, defined by

$$\mathcal{M}_u:=\left\{j:\prod_{n\in\mathcal{N}}|\lambda_j^{(n)}|^{\pi_n}>1\,,j=1,2,...,m\right\},$$

$$\mathcal{M}_s:=\{1,...,m\}\setminus\mathcal{M}_u\,.$$

Corollary 4.9: Suppose that the Lie algebra $\mathfrak{g}$ is *solvable*. Then we have

$$\bar{I}((\mathbf{x}_0,\boldsymbol{\sigma});\mathbf{e})\geq\sum_{n\in\mathcal{N}}\sum_{j\in\mathcal{M}_u}\pi_n\log|\lambda_j^{(n)}|$$

Proof: We start with a mutually disjoint partition of the index set $\{1,2,...,\boldsymbol{\sigma}(k)\}$, given by

$$\{1,2,...,\boldsymbol{\sigma}(k)\}=\bigcup_{n\in\mathcal{N}}\mathcal{K}_n\,,$$

where $\mathcal{K}_n:=\{i:\boldsymbol{\sigma}(i)=n,i=1,2,..,k\}$. Then we claim that the eigenvalues of F_k take the form $\lambda_j(F_k)=\prod_{n\in\mathcal{N}}\prod_{j=1}^m\left(\lambda_j^{(n)}\right)^{|\mathcal{K}_n|}$, where $\lambda_j^{(n)}$ is the diagonal entry from (7). Indeed, it is easy to see that $T^{-1}F_kT=T^{-1}A(\boldsymbol{\sigma}(k))TT^{-1}A(\boldsymbol{\sigma}(k-1))T\cdots T^{-1}A(\boldsymbol{\sigma}(0))T$ is a triangular matrix for all k. Further, the jth diagonal entry of $T^{-1}F_kT$ can be calculated as

$$\lambda_j(F_k)=\prod_{i=0}^{k}\lambda_j^{(\boldsymbol{\sigma}(i))}=\prod_{n\in\mathcal{N}}\left(\lambda_j^{(n)}\right)^{|\mathcal{K}_n|}$$

Using the above relation and Fatou's Lemma we have

$$\begin{aligned}&\liminf_{k\to\infty}\frac{1}{k+1}\mathbf{E}\sum_j\Re\left(\log\lambda_j\left(F_k\right)\right)^+\\&=\liminf_{k\to\infty}\mathbf{E}\frac{1}{k+1}\sum_j\Re\left(\log\lambda_j(F_k)\right)^+\\&\geq\mathbf{E}\liminf_{k\to\infty}\frac{1}{k+1}\sum_j\Re\left(\log\lambda_j(F_k)\right)^+.\end{aligned}$$

Furthermore,

$$
\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{k+1} \sum_j \Re\left(\log \lambda_j(F_k)\right)^+ \\
&= \lim_{k\to\infty} \frac{1}{k+1} \sum_j \Re\left(\log \lambda_j(F_k)\right)^+ \\
&= \sum_j \Re\left(\sum_n \pi_n \log \lambda_j^{(n)}\right)^+ \\
&= \sum_{n\in\mathcal{N}} \sum_{j\in\mathcal{M}_u} \pi_n \log |\lambda_j^{(n)}| \,,
\end{aligned}
$$

where the second equality follows from ergodicity of $\boldsymbol{\sigma}(k)$ and the Law of Large Numbers [28]. ■

Remark 4.10: As explained in [5], this modern algebraic approach, though mathematically appealing, shows a significant drawback for its lack of robustness, i.e. even a very small perturbation of system parameters can violate the solvability condition. One may conduct perturbation analysis to relax the algebraic structure requirement, though it is not trivial in general.

C. Bode's Integral

Theorem 4.11: Consider the closed loop in Fig. 1. If $\mathbf{d}$ and $\mathbf{e}$ form Gaussian stationary processes, then

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega \geq \liminf_{k\to\infty} \frac{1}{k+1} \sum_i \Re\left(\log \lambda_i\left(F_k\right)\right)^+ \,.
$$

Proof: This result is evident by considering the following relation, followed by Kolmogrov's formula

$$
\bar{h}(\mathbf{d}) = \log(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log f_{\mathbf{d}}(\omega) d\omega \,,
$$

$$
\bar{h}(\mathbf{e}) = \log(2\pi e) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log f_{\mathbf{e}}(\omega) d\omega \,,
$$

together with Theorem 4.6. ■

Since we have obtained various lower bounds for $\bar{I}(\mathbf{x}_0, \mathbf{d}, \boldsymbol{\sigma}; \mathbf{e})$ in the previous subsection, the following corollaries can be readily obtained.

Corollary 4.12: Consider the closed loop in Fig. 1. If $\mathbf{d}$ and $\mathbf{e}$ form Gaussian stationary processes, then

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega \geq \log \prod_{n\in\mathcal{N}} |\det A(n)|^{\pi_n} \,.
$$

Corollary 4.13: Consider the closed loop in Fig. 1. If $\mathbf{d}$ and $\mathbf{e}$ form Gaussian stationary processes, and the Lie algebra $\mathfrak{g}$ is *solvable*, then

$$
\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega \geq \sum_{n\in\mathcal{N}} \sum_{j\in\mathcal{M}_u} \pi_n \log |\lambda_j^{(n)}| \,.
$$

Remark 4.14: Similar to its deterministic counterpart, Bode's integral in this stochastic setting also captures the performance limitation of a closed loop in frequency domain. The lower bound of the achievable performance is inherent from its open loop plant instability.

Remark 4.15: Though it is hard to determine whether the closed loop in Fig. 1 is stationary in general, some results for LTI systems can be found in [7] and [29].

D. Data Rate Inequality

The next theorem provides a lower bound for the mutual information rate $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$, which accounts for total information rate flow in a causal loop. Further insight into $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$ can be found in [2] and [30].

Theorem 4.16: Consider the closed-loop system shown in Fig. 1. We have the following inequality:

$$\bar{I}((\mathbf{x}_0, \mathbf{d}, \boldsymbol{\sigma}); \mathbf{u}) \geq \bar{I}(\mathbf{x}_0, \boldsymbol{\sigma}; \mathbf{e}) + \bar{I}(\mathbf{d}; \mathbf{u})\,.$$

Proof: Using Kolmogorov's formula (P2), we have

$$I((\mathbf{x}_0, \mathbf{d}^k, \boldsymbol{\sigma}^k); \mathbf{u}^k) = I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{u}^k | \mathbf{d}^k) + I(\mathbf{u}^k; \mathbf{d}^k)\,,$$

where $k \in \mathbb{N}^+$ is an arbitrary time instance. We can lower bound $I((\mathbf{x}_0, \mathbf{d}^k); \mathbf{u}^k)$ as

$$\begin{aligned} & I((\mathbf{x}_0, \boldsymbol{\sigma}^k, \mathbf{d}^k); \mathbf{u}^k) \\ & \overset{(a)}{=} I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k | \mathbf{d}^k) + I(\mathbf{u}^k; \mathbf{d}^k) \\ & \overset{(b)}{=} I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) - I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{d}^k) + \\ & \qquad\qquad I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{d}^k | \mathbf{e}^k) + I(\mathbf{u}^k; \mathbf{d}^k) \\ & \overset{(c)}{=} I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) + I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{d}^k | \mathbf{e}^k) + I(\mathbf{u}^k; \mathbf{d}^k) \\ & \overset{(d)}{\geq} I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) + I(\mathbf{u}^k; \mathbf{d}^k)\,. \end{aligned}$$

Here (a) follows from the fact that $I(\mathbf{x}_0; \mathbf{u}^k | \mathbf{d}^k) = I(\mathbf{x}_0; \mathbf{u}^k + \mathbf{d}^k | \mathbf{d}^k) = I(\mathbf{x}_0; \mathbf{e}^k | \mathbf{d}^k)$; (b) follows from (P2); (c) follows from the independence of $\mathbf{d}$ and $\mathbf{x}_0$; and (d) follows from the fact that $I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{d}^k | \mathbf{e}^k) \geq 0$. We have obtained the following inequality:

$$I((\mathbf{x}_0, \mathbf{d}^k, \boldsymbol{\sigma}^k); \mathbf{u}^k) \geq I(\mathbf{x}_0, \boldsymbol{\sigma}^k; \mathbf{e}^k) + I(\mathbf{u}^k; \mathbf{d}^k)\,. \tag{8}$$

The conclusion is readily obtained by dividing the terms on both sides of (8) by $k+1$ and taking the limit as $k \to \infty$. ■

E. Bound Tightening via Lifting

All the above obtained results are based on *one-step* behavior of the Markov chain $\boldsymbol{\sigma}$, however, simple observation shows that Markov property and ergodicity also hold for the *multi-step* version of $\boldsymbol{\sigma}$ (denoted as $\boldsymbol{\sigma}^m$ for m step), and the multi-step dynamics retains the same performance limitation as in the one-step case. More precisely, we can find a lifted representation of the original system that describes the system dynamics over a lifting horizon m. The lifted dynamics can be then defined as a new switched system, where the mode becomes m-dimensional Cartesian product of the original index set (therefore the number of new modes becomes N^m). We can then calculate the stationary distribution over the new modes, and proceed with previous analysis to obtain the lower bound on the performance. Since only the long-term behavior is considered for the performance limitation, all the m-lifted results also hold for the original dynamics. In other words, we consider the lifted dynamics for its degree of instability, which may potentially improve the bounds on the performance limitation of the closed loop of interest.

Given the lift horizon m we have the m-lifted dynamics of the the open-loop plants

$$\tilde{\mathbf{x}}(h+1) = \tilde{A}(\tilde{\boldsymbol{\sigma}}(h))\tilde{x}(h)\,, \tag{9}$$

where $\tilde{\mathbf{x}}(h) = \mathbf{x}(mh)$, $\tilde{\boldsymbol{\sigma}}(h) = (\boldsymbol{\sigma}(mh), ..., \boldsymbol{\sigma}(mh+m-1))$ and $\tilde{A}(\tilde{\boldsymbol{\sigma}}(h)) = A(\boldsymbol{\sigma}(mh+m-1))...A(\boldsymbol{\sigma}(mh))$. For this switched dynamics we can obtain an invariant distribution $\tilde{\boldsymbol{\pi}}$, which is an

N^m vector. The next theorem bridges the lifted system and the original system in terms of the degree of instability.

Theorem 4.17: Consider the original closed loop. Then the following inequality holds

$$\bar{I}((\mathbf{x}_0, \boldsymbol{\sigma}); \mathbf{e}) \geq \liminf_{h\to\infty} \frac{1}{h+1} \mathbf{E} \sum_j \Re \left(\log \lambda_j \left(\tilde{F}_h \right) \right)^+ , \tag{10}$$

where $F_h := A(\tilde{\boldsymbol{\sigma}}(h))A(\tilde{\boldsymbol{\sigma}}(h-1)) \cdots A(\tilde{\boldsymbol{\sigma}}(0))$.

Proof: The proof follows the same steps in the proof for Theorem 4.6, and therefore is omitted here. ■

The above theorem implies that one can replace the original Lyapunov exponent by a lifted version, which may give a better estimation of the performance bound. Moreover, starting with this theorem, the algebraic conditions can be tailored to the lifted case, so as the ultimate Bode's formula. The detailed development is not given here, as it essentially repeats what has been done for the one-step case in this section.

Remark 4.18: This approach is greatly inspired by [31], where lift is used to obtain a better stability criterion for jump linear systems.

V. Example: Networked Control Systems with Random Packet Dropouts

In this section, we apply the framework from the previous section to examine the performance limitation problems in networked control systems (NCS). To be specific, we only consider the control systems with a lossy communication channel placed between the sensor and the controller, which has been studied in various papers [21] [12] [32]. In this paper we adopt a structure similar to [12], shown in Fig. 2, where a switch is placed after the output to model packet dropouts.

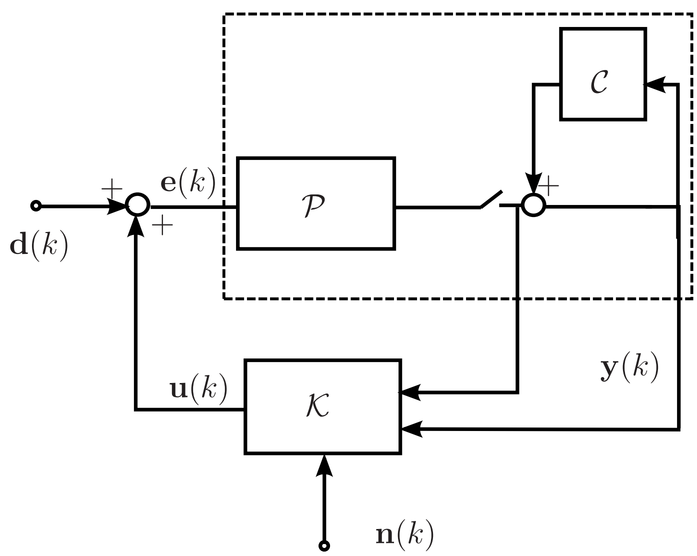

Fig. 2. A networked control system

The packet dropouts are compensated for by an output of an LTI system, which has to be designed. The controller can be represented by any causal map from $\mathbf{y}_0^k$ to $u(k)$. The sequence of *ON's* and *OFF's* of the erasure channel is modeled as a two-state Markov chain with transition probability matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}, \qquad 0 \leq p, q \leq 1 \,.$$

One can calculate the stationary distribution as $\boldsymbol{\pi} = \left[\frac{q}{p+q}, \frac{p}{p+q}\right]$. Let the state space realization of the plant and the channel compensator be

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \text{ and } \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$$

respectively. We can then regard the dashed box in Fig. 2 as a generalized "plant" with state matrices $\tilde{A}(1) = \begin{bmatrix} A & 0 \\ B_cC & A_c \end{bmatrix}$, $\tilde{A}(2) = \begin{bmatrix} A & 0 \\ 0 & A_c + B_cC_c \end{bmatrix}$ for the "ON" and "OFF" of the erasure channel respectively.

To simplify the subsequent analysis, we further assume that the compensator is chosen such that A_c and $A_c + B_cC_c$ are stable, which is the case in [12] and [21]. Under these additional conditions and with account of Theorem 4.6 we have the corresponding Bode's integral theorem.

Theorem 5.1: Consider the NCS in Fig. 2, and assume that the signal $\mathbf{u}$ is Gaussian and stationary. The following relation holds for all causal controllers $\mathcal{K}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega \geq \sum_j \Re\left(\log \lambda_j\left(A\right)\right)^+ . \tag{11}$$

Proof: The proof is a simple application of Theorem 4.11, and is therefore omitted here. ∎

Remark 5.2: This theorem characterizes the control design limitation for NCS with random packet dropout. Given the stable compensator, the right hand side in (11) shows that the lower bound of the closed loop performance is determined solely by the degree of instability of A. This observation suggests that, considering the relatively loose definition of stability in (1), packet dropout does not make the system "more" unstable. However, the dropout may add up to the performance limitation in other forms, for which a close scrutiny is required.

VI. Conclusions

This paper developed a relatively complete Bode's integral formula for a stochastic switched closed loops. Information theory has been employed as a tool to obtain a relationship among different system variables, which has in turn resulted in Bode's integral for stationary cases. Various algebraic conditions have been proposed to capture tight performance bounds. An example of applying this theoretic framework to the field of NCS illustrates the usefulness of this fundamental result.

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