Bode-like Integral for Continuous-Time Closed-Loop Systems in the Presence of Limited Information

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Abstract

This paper analyzes causal closed-loop continuous-time systems in the presence of limited information. Assuming that the exogenous signals can be modeled as a stochastic process, a mutual information rate inequality is obtained that can be viewed as an extended Bode-type formula for stationary processes. The tightness of the resulting Bode's integral inequality is then analyzed for the linear time invariant closed loops. Within the developed framework we consider the control–communication interplay and analyze the underlying fundamental limitations.

I. Introduction

Recent progress in communication technologies and their use in feedback control systems motivate to look deeper into the interplay of control and communication in the closed-loop feedback architecture. Among several research directions on this topic, a great deal of attention has been given to the fundamental limitations of feedback control in the presence communication channels. An important observation was made in [1] that limited communication of control signals can prevent controllers to achieve sufficient accuracy leading possibly to instability, and a critical bound on the communication limitation (channel capacity) was derived towards obtaining closed-loop stability. Further investigation conducted in [2]–[5] showed that the LTI plants can be stabilized in deterministic or stochastic manner via finite-alphabet channels with proper decoding

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and encoding schemes. In the case of noisy channels, the notion of *anytime capacity* is introduced to cope with unstable information sources, which are not treated in Shannon's theory. As an effort to broaden the class of problems from stabilization to more general cases such as disturbance rejection, [6] and [7] have presented a fairly general and complete approach in discrete-time setting towards unification of information theory and control theory. Entropy rate inequalities corresponding to the information flux in a typical causal closed loop have been derived towards obtaining a Bode-like integral formula. The extensions of Bode's theorem have been claimed for certain discrete-time nonlinear systems and linear time-varying systems respectively, [8], [9], and later for nonlinear ones [10].

We notice that most of the prior results in this direction are derived for discrete-time systems. In this paper we investigate the continuous-time systems for the following reasons. First, a large number of real-life plants are continuous-time in nature, and therefore it is of interest to develop the corresponding continuous-time tools for closed-loop analysis. While discretization provides a popular practice for control designs, it is also widely understood that the sampled dynamics over countable indices may under-represent their continuous-time counterparts defined on the time continuum. Among many proven examples, in the adaptive control field, [11] reveals the fundamental differences between adaptive control of continuous- and discrete-time systems with nonlinear growing nonlinearity: a globally stable adaptive controller can be only designed for continuous-time but not for discrete-time systems in general. Second, although digital channels dominate almost all communication systems, some continuous-time models such as continuous-time Additive Gaussian White Noise (AWGN) channels catch significant attention because of their theoretical simplicity but non-triviality [12], [13]. From technique point of view, the continuous-time case of Bode's integral imposes challenges for both control theory and information theory. As for control, except for the classical Bode's result and its extensions [14], where Bode's integral formulae for continuous-time and discrete-time systems are bridged by Poisson's integral formula, there is no similar mathematical tool available yet in the general setting. As for information theory, we point out that the results in [7] and [6], together with several others [15]–[17], heavily rely upon the following entropy rate equality originated by Kolmogorov [18]:

$$\bar{h}(\xi) = \log(2\pi\sqrt{e}) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f_\xi(\lambda) d\lambda \,, \tag{1}$$

where ξ is a Gaussian discrete-time stationary process, $\bar{h}$ stands for its entropy rate, and f_ξ is the spectral density function of ξ. This formula, however, is only applicable to discrete-time processes, and its continuous-time extension *has to be derived otherwise* [19]. However, no such extension has been carried out since Kolmogrov's comment because of the undesirable behavior of differential Shannon entropy rate for continuous-time processes.

In this paper, we attempt to use tools from information theory to analyze performance limitations for continuous-time systems with stochastic disturbances. We first derive the mutual information rate inequality by assuming causality of the closed-loop system. A Bode-type formula is then obtained to address the fundamental limitation of the stabilization problem in frequency domain. The techniques utilized here are different from discrete-time case in that: $i)$ mutual information rate instead of entropy rate is adopted to represent the information flow in a closed-loop; $ii)$ to get the Bode-type integral, we use the result from [20], which helps to circumvent Kolmogorov's formula (1). To get the insight into the resulting Bode's integral, we then employ tools from complex analysis to identify the extra term of performance limitation induced by the controller/channel noise. We also quantify the negative portion of the Bode's integral and relate it to the rate of information passing through the closed loop. Finally we apply this framework to communication–control interconnection to study the relationship between the channel capacity and the stability of the closed-loop systems.

During recent years, several results have been reported for continuous-time systems with limited information. In [21] necessary and sufficient conditions for stabilizability and observability of LTI systems over a class of additive Gaussian channels have been provided, together with continuous time encoding and decoding scheme for closed-loop control. In [22] and [23], a method of obtaining a tight upper bound on the SNR, based on $\mathcal{H}_2$-control type argument, has been developed. In [24] a time-domain Bode's integral has been proposed for both LTV and LTI systems by examining the dynamics of the plant and an extension of Szegö's limit theorem. A design methodology for channel noise attenuation has been reported recently in [25].

The paper is organized as follows. In Section II, we introduce the closed-loop feedback configuration and some basic definitions and facts from information theory and stochastic processes. Section III studies a general feedback scheme, within which we develop a mutual information inequality and a Bode-type integral formula. Section IV further explores the relation of Bode's integral with the information transmission rate of the closed loop, while Section V carries out the in-depth analysis of the the Bode-type integral by using complex integration techniques. The paper is concluded in Section VII. We note that Sections V, IV and VI are developed in somewhat parallel manner, and the reader should not be surprised to find forward cross-referencing among these sections.

Notation:

- $\mathbb{R}$ denotes the field of real numbers; $\mathbb{C}$ stands for complex plane; $\mathbb{C}^-$ and $\mathbb{C}^+$ stand for the left half and right half of $\mathbb{C}$ respectively.
- Random variables defined in appropriate probability spaces are represented using boldface letters, such as $\mathbf{x}$, $\mathbf{y}$. If not otherwise stated, the random variables take values in $\mathbb{R}$ throughout the paper.
- If $\mathbf{x}(k), k \in \mathbb{N}^+$, is a discrete time stochastic process, we denote its segment $\{\mathbf{x}(k)\}_{k=l}^{u}$ by $\mathbf{x}_l^u$, and use $\mathbf{x}_0^n := \mathbf{x}^n$ for simplicity.
- Consider a continuous time stochastic process $\mathbf{x}(t), t \in \mathbb{R}^+$. A sample path on an interval $[t_1, t_2],\ 0 \leq t_1 < t_2 \leq +\infty$, is indicated as $\mathbf{x}_{t_1}^{t_2}$. We also denote $\mathbf{x}_0^t := \mathbf{x}^t$ for simplicity.
- $\mathbf{x}^{(h)}$ is the discrete-time process obtained from sampling of $\mathbf{x}(t)$ on $t \in [t_1, t_2]$ with an interval $h > 0$. We denote $\mathbf{x}_i^{(h)} = \mathbf{x}^{(h)}(i) := \mathbf{x}(t_1 + ih), \quad i = 0, 1, ...$
- The probability density (if it exists) of a random variable $\mathbf{x}$ is represented as $p_{\mathbf{x}}$.
- $\mathbf{E}[\cdot]$ is the expectation operator of a random variable.
- $(\cdot)^+ = \max\{\cdot, 0\}$ and $(\cdot)^- = \min\{\cdot, 0\}$.
- $\Re(\cdot)$ gives the real part of a complex number.
- $\lambda_i(\cdot)$ gives the eigenvalues of a square matrix.
- $Re(\cdot; z)$ gives the residue of an analytical function at $z \in \mathbb{C}$.
- $h(\cdot)$ and $h(\cdot|\cdot)$ stand for (differential) entropy and conditional entropy respectively and $I(\cdot;\cdot)$ and $I(\cdot;\cdot|\cdot)$ stand for mutual information and conditional mutual information respectively.

II. Preliminaries

In this section, several basic definitions and related facts from information theory and stochastic processes are introduced. We rely on [26] an [27] as main references.

A. Entropy, Mutual Information and Related Facts

In this subsection, we introduce some elementary definitions and results from information theory, most of which are taken from [26].

Definition 2.1: [Entropy Rate] The entropy rate of $\mathbf{x}$ is defined as

$$\bar{h}(\mathbf{x}) := \lim_{n\to\infty} \frac{h(\mathbf{x}^n)}{n+1}\,, \tag{2}$$

given the existence of the limit.

Definition 2.2 (Mutual Information Rate): The mutual information rate of two stochastic processes is defined as

$$\bar{I}(\mathbf{x};\mathbf{y}) := \lim_{n\to\infty} \frac{I(\mathbf{x}^n;\mathbf{y}^n)}{n+1}\,, \tag{3}$$

given the existence of the limit.

To consider the information between two continuous-time stochastic processes we introduce the following definition [20].

Definition 2.3 (Mutual Information of Continuous Processes): The mutual information between two stochastic processes $\mathbf{x}$ and $\mathbf{y}$ on time interval $[s,t)$, $0 \le s \le t < \infty$, is defined as

$$I(\mathbf{x}_s^t;\mathbf{y}_s^t) := \int \log \frac{dP_{\mathbf{x}_s^t,\mathbf{y}_s^t}}{dP_{\mathbf{x}_s^t} \times dP_{\mathbf{y}_s^t}} dP_{\mathbf{x}_s^t,\mathbf{y}_s^t}\,, \tag{4}$$

where $P_{\mathbf{x}_s^t}$, $P_{\mathbf{y}_s^t}$ and $P_{\mathbf{x}_s^t,\mathbf{y}_s^t}$ are the probability measures, induced by random objects $\mathbf{x}_s^t$, $\mathbf{y}_s^t$ and $(\mathbf{x}_s^t,\mathbf{y}_s^t)$ respectively, and $\frac{dP_{\mathbf{x}_s^t,\mathbf{y}_s^t}}{dP_{\mathbf{x}_s^t}\times dP_{\mathbf{y}_s^t}}$ is the Radon-Nikodym derivative, given that $P_{\mathbf{x}_s^t,\mathbf{y}_s^t}$ is absolutely continuous with respect to the product measure $P_{\mathbf{x}_s^t} \times P_{\mathbf{y}_s^t}$.
Similar to Definition 2.2, we define the *information rate* for continuous-time processes.

Definition 2.4 (Information Rate): The information rate is given by

$$\bar{I}(\mathbf{x};\mathbf{y}) := \lim_{T\to\infty} \frac{I(\mathbf{x}^T;\mathbf{y}^T)}{T}\,, \tag{5}$$

given the existence of the limit.
In (5), $\bar{I}$ could be viewed as the rate of mutual information for reliable transmission through any communication channel ($\mathbf{x}$ as input and $\mathbf{y}$ as output or vice versa).

Remark 2.5: It is worth mentioning that, according to convention, we avoid the notion of differential entropy $h(\cdot)$ for a segment of a continuous time process, because h can be infinite for certain processes. In turn, the definition “entropy rate” $\bar{h}(\cdot)$ in continuous-time setting has been precluded. To take a glimpse at the root of this discrepancy between discrete- and continuous-time, one is reminded of the well-known fact that *differential entropy* in general does not quantify the actual "information" retaining by an $\mathbb{R}$-defined random variable as opposed to *entropy*, which is defined for finite-alphabet random variables [26].

The next lemma gives the opportunity to represent the continuous time mutual information as the limit of its discretized version.

Lemma 2.6: Consider continuous-time stochastic processes $\mathbf{x}$ and $\mathbf{y}$. The mutual information between $\mathbf{x}_s^t$ and $\mathbf{y}_s^t$, $0 \leq s < t < \infty$, can be obtained as

$$\begin{aligned} & I(\mathbf{x}_s^t;\mathbf{y}_s^t) = \lim_{n\to\infty} I(\mathbf{x}_0^{(\delta(n))},...,\mathbf{x}_n^{(\delta(n))};\mathbf{y}_0^{(\delta(n))},...,\mathbf{y}_n^{(\delta(n))})\,, \\ & \mathbf{x}_i^{(\delta(n))} = \mathbf{x}(s+i\delta(n))\,, i=0,1,... \end{aligned} \tag{6}$$

for any fixed s and t with $\delta(n) = \frac{t-s}{n+1}$.

The proof of this lemma is a mere application of the result in [28] on the interval $[s,t]$, which can be viewed as a 1-D closed manifold and could be represented as the closure of all the sampling points.

This lemma is used successfully in [29] to connect discrete-time results with continuous-time ones regarding the channel sensitivity. The inherent sampling type of argument in the lemma *permits the general information measures to inherent many of its properties from the simpler discrete-time case* [30]. It will also serve as an important tool to obtain the main result in this work.

A list of useful properties of entropy and mutual information are given here, and are frequently used in the upcoming arguments.

(P1) *Symmetry and nonnegativity*:

$$I(\mathbf{x};\mathbf{y}) = I(\mathbf{y};\mathbf{x}) = h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) \geq 0\,.$$

(P2) *Kolmogorov equality*:

$$I(\mathbf{x};(\mathbf{y},\mathbf{z})) = I(\mathbf{x};\mathbf{z}) + I(\mathbf{x};\mathbf{y}|\mathbf{z})$$

(P3) *Data processing inequality*:

$$I(\mathbf{x};\mathbf{y}|\mathbf{z}) \geq I(\mathbf{x};g(\mathbf{y})|\mathbf{z})$$

The equality holds, if $g(\cdot)$ is invertible and the inversion $g^{-1}(\cdot)$ is measurable.

(P4) *Invariance of mutual information (entropy)*

$$I(\mathbf{x};\mathbf{y}|\mathbf{z}) = I(\mathbf{x}+g(\mathbf{z});\mathbf{y}|\mathbf{z})\,, h(\mathbf{x}|\mathbf{z}) = h(\mathbf{x}+g(\mathbf{z})|\mathbf{z}),$$

where $g(\cdot)$ is a function.

(P5) *Chain rule*:

$$h(\mathbf{x}^n|\mathbf{y}) = \sum_{k=1}^{n} h(\mathbf{x}_k|\mathbf{y},\mathbf{x}^{k-1})$$

(P6) *Maximum entropy*: Consider $\mathbf{x} \in \mathbb{R}^m$ and the covariance matrix given by $V := \mathbf{E}[\mathbf{x}\mathbf{x}^\top]$. Then we have

$$h(\mathbf{x}) \leq h(\bar{\mathbf{x}}) = \frac{1}{2}\log((2\pi e)^m \det V)\,,$$

where $\bar{\mathbf{x}}$ is a Gaussian process with the same covariance as $\mathbf{x}$. Equality holds, if $\mathbf{x}$ is Gaussian.

B. Spectral Analysis of Stationary Stochastic Processes

Here we introduce some results related to the spectral theory of stationary processes.

Definition 2.7 (Wide Sense Stationary Process): A zero-mean continuous-time stochastic process $\mathbf{x}(t) \in \mathbb{R}^n$, $t \geq 0$, is stationary, if for all $t \geq 0$ its covariance function, defined by

$$R_{\mathbf{x}}(\tau) = \mathbf{E}[\mathbf{x}(t+\tau)\mathbf{x}^\top(t)], \quad \tau \in \mathbb{R}\,, \tag{7}$$

is independent of t. Throughout this paper, *wide sense stationary* is abbreviated as *stationary* for convenience.

The spectral decomposition of the covariance function $R_{\mathbf{x}}(t)$ is defined via Fourier transform:

$$f_{\mathbf{x}}(\omega) = \int_0^\infty e^{-it\omega} R_{\mathbf{x}}(t)dt\,, \tag{8}$$

and the function $f_{\mathbf{x}}(\cdot)$ is called *power spectral density (PSD)* of $\mathbf{x}$. The stationary process $\mathbf{x}$ admits a *spectral factorization*, if

$$f_{\mathbf{x}}(\omega) = \phi_{\mathbf{x}}(-j\omega)\phi_{\mathbf{x}}(j\omega)\,,$$

for some function $\phi_{\mathbf{x}}(\cdot)$. The following lemma from [31] shows that a rational PSD always admits a rational spectral factorization.

Lemma 2.8: If $f_{\mathbf{x}}(\omega)$ is rational, then there exits a minimum phase and asymptotically stable LTI system $\phi_{\mathbf{x}}(s)$, such that

$$f_{\mathbf{x}}(\omega) = \phi_{\mathbf{x}}(-j\omega)\phi_{\mathbf{x}}(j\omega)$$

There are various ways to find $\phi_{\mathbf{x}}$; the reader is referred to [32] for an extensive overview. We define class $\mathbb{F}$ functions as follows [33].

Definition 2.9 (Class $\mathbb{F}$ function):

$$\mathbb{F} = \{l : l(\omega) = p(\omega)(1 - \varphi(\omega)), l(\omega) \in \mathbb{C}, \omega \in \mathbb{R}\}\,, \tag{9}$$

where $p(\cdot)$ is rational and $\varphi(\cdot)$ is a measurable function, such that $0 \leq \varphi < 1$ for all $\omega \in \mathbb{R}$ and $\int_{\mathbb{R}} |\log(1 - \varphi(\omega))| d\omega < \infty$.

It is obvious that all rational functions are in $\mathbb{F}$.

The following lemma is taken from [20], which gives a lower bound on the mutual information rate of two continuous-time Gaussian stationary processes.

Lemma 2.10: Suppose that two one-dimensional continuous-time processes $\mathbf{x}$ and $\mathbf{y}$ form a stationary Gaussian process $(\mathbf{x}, \mathbf{y})$. Then

$$\bar{I}(\mathbf{x}, \mathbf{y}) \geq -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log\left(1 - \frac{|f_{\mathbf{xy}}(\omega)|^2}{f_{\mathbf{x}}(\omega) f_{\mathbf{y}}(\omega)}\right) d\omega\,. \tag{10}$$

The equality holds, if $f_{\mathbf{x}}$ or $f_{\mathbf{y}}$ belong to the class $\mathbb{F}$.

C. Closed-Loop System

Throughout the paper we consider the feedback configuration depicted in Fig. 1.

Several assumptions are made:

- The plant $\mathcal{P}$ is modeled by the following stochastic differential equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{e}(t)\,, \quad \mathbf{x}(0) = \mathbf{x}_0\,, \\ \mathbf{y}(t) &= C\mathbf{x}(t)\,. \end{aligned} \tag{11}$$

  Here $\mathbf{x}(t) \in \mathbb{R}^n$, and $\mathbf{x}_0$ is assumed to have finite differential entropy or $|h(\mathbf{x}_0)| < \infty$.
- An arbitrary small time-delay $\epsilon > 0$ is imposed on the output signal $\mathbf{y}$ as a technical assumption.
- The disturbance $\mathbf{d}(t)$ is a stochastic process, and $\mathbf{n}(t)$ is a stochastic process that models the implicit controller noise. We assume that $\mathbf{d}(t)$, $\mathbf{n}(t)$ and $\mathbf{x}_0$ are mutually independent.

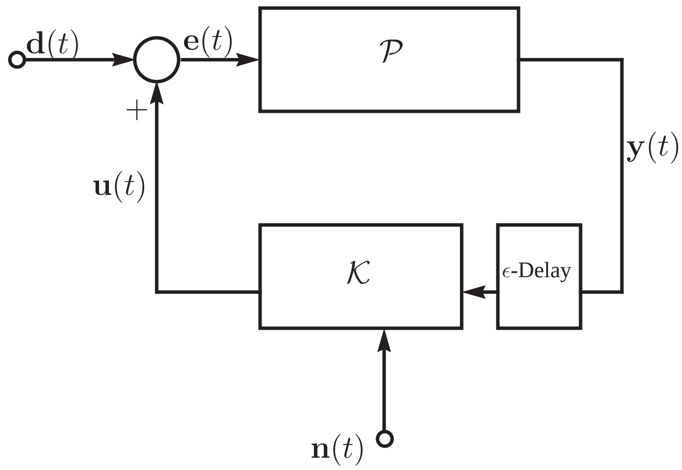

Fig. 1. Basic Feedback Scheme

- The generalized controller $\mathcal{K}$ is given as a causal map such that

$$\mathcal{K} : (\mathbf{y}_0^{t-\epsilon}, \mathbf{n}_0^t) \mapsto \mathbf{u}(t)\,.$$

The above closed-loop configuration, bearing a close resemblance to its discrete-time counterpart in [7], is introduced to represent a broad class of control systems with stochastic disturbances and information constraints. The generalized controller $\mathcal{K}$ may contain the control algorithm together with the information transmission scheme. As it appears later, $\mathbf{n}(t)$ can be versatilely used to model observation noise, channel noise or output disturbance, dependent on the realization of $\mathcal{K}$. The assumption of ϵ delay is the continuous time version of "one step" delay ubiquitous in the discrete time literature, which is also justifiable from the fact that any physical machinery is subject to time-delay. To keep the problem formulation sufficiently general, the distributions of stochastic processes $\mathbf{d}(t)$ and $\mathbf{n}(t)$ and random variable $\mathbf{x}_0$ are not assumed, although further assumptions on their nature will be stated in some of the results in this paper.

Definition 2.11 (Sensitivity-like Function): A sensitivity-like function of the closed loop is defined as

$$S_{\mathbf{d},\mathbf{e}}(\omega) = \sqrt{\frac{f_{\mathbf{e}}(\omega)}{f_{\mathbf{d}}(\omega)}}\,, \tag{12}$$

where $\mathbf{e}$ and $\mathbf{d}$ are stationary and stationarily correlated.

Remark 2.12: The function $S_{\mathbf{d},\mathbf{e}}(\omega)$ is the stochastic analogue of the sensitivity function $|S(j\omega)|$ in Bode's original work [34].

Throughout, we adopt the following stability definition.

Definition 2.13 (Mean-square Stability): The closed loop given in Fig. 1 is said to be mean-square stable, if

$$\sup_{t\geq 0} \mathbf{E}[\mathbf{x}^\top(t)\mathbf{x}(t)] < \infty \,. \tag{13}$$

III. Information Conservation Law and Extension of Bode's Integral Formula

As it has been revealed in [7], causality plays a central role in obtaining a Bode-type formula for a discrete-time feedback loop with stochastic disturbance. Bearing this observation in mind, we then obtain a set of mutual information rate inequalities resulting directly from the feedback structure and causality of the closed loop shown in Fig 1. In turn, an analogue of Bode's theorem is obtained by assuming certain stationarity property for the disturbance signal.

To start with, we introduce the following Lemma, where the sum of all the unstable eigenvalues (or the degree of instability) of the open loop state matrix A is upper bounded by the mutual information rate between the initial value $\mathbf{x}_0$ and the error signal $\mathbf{e}$, given that the closed loop is stable.

Lemma 3.1: If the closed-loop system in Fig. 1 is stable, then the following inequality holds

$$\bar{I}(\mathbf{x}_0; \mathbf{e}) \geq \sum_i \Re(\lambda_i(A))^+ \,, \tag{14}$$

where $\Re(\lambda_i(A))^+ := \max\{0, \Re(\lambda_i(A))\}$.

Proof: If A is Hurwitz, then $\sum_i \Re(\lambda_i(A))^+ = 0$ and (14) trivially holds. In case A is not Hurwitz, one can decompose A, by a nonsingular matrix G, into A_s and A_u with stable and unstable eigenvalues respectively. Accordingly, the state $\mathbf{x}(t)$ can be represented as $\mathbf{x}(t) = G[\mathbf{x}_s^\top(t), \mathbf{x}_u^\top(t)]^\top$, where $\mathbf{x}_s$ and $\mathbf{x}_u$ indicate the stable and unstable sub-state vectors respectively. We then consider the following unstable dynamics:

$$\dot{\mathbf{x}}_u(t) = A_u\mathbf{x}_u(t) + B_u\mathbf{e}(t) \,, \tag{15}$$

where B_u stands for the submatrix of $G^{-1}B$ corresponding to A_u. The solution to (15) is written as

$$\begin{aligned}\mathbf{x}_u(t) &= \exp(A_u t)\mathbf{x}_u(0) + \int_0^t \exp(A_u(t-\tau))B_u\mathbf{e}(\tau)d\tau \\ &= \exp(A_u t)\left(\mathbf{x}_u(0) + \int_0^t \exp(-A_u\tau)B_u\mathbf{e}(\tau)d\tau\right) \\ &= \exp(A_u t)(\mathbf{x}_u(0) + \hat{\mathbf{x}}_u(t)) \quad \forall\ t > 0\,,\end{aligned} \tag{16}$$

where we have defined

$$\hat{\mathbf{x}}_u(t) := \int_0^t \exp(-A_u\tau)B_u\mathbf{e}(\tau)d\tau\,.$$

The mean-square stability condition in (13) implies that for all t, each entry of matrix $\mathbf{E}\mathbf{x}^\top(t)\mathbf{x}(t)$ is finite. To see this, one can simply express each entry as $\mathbf{E}\mathbf{x}_i(t)\mathbf{x}_j(t)$ ($\mathbf{x}_i$ and $\mathbf{x}_j$ stand for i^{th} and j^{th} entries of vector $\mathbf{x}$ respectively), whose absolute value can be easily upper-bounded by Cauchy-Schwartz inequality $|\mathbf{E}\mathbf{x}_i(t)\mathbf{x}_j(t)| \leq \sqrt{\mathbf{E}\mathbf{x}_i^2(t)\mathbf{E}\mathbf{x}_j^2(t)} < +\infty$. The same fact holds for $\mathbf{E}\mathbf{x}_u^\top(t)\mathbf{x}_u(t)$, since $\mathbf{x}_u$ is a sub-vector of linearly transformed $\mathbf{x}$. It is then easy to see that $|\det \mathbf{E}\mathbf{x}_u^\top(t)\mathbf{x}_u(t)| < +\infty$. Further, from the monotonicity of $\log(\cdot)$ we have

$$\begin{aligned}+\infty > M > \log\left(\det \mathbf{E}(\mathbf{x}_u(t)\mathbf{x}_u^\top(t))\right) &= 2t\log\left(\det(\exp(A_u))\right) \\ &+ \log\left(\det \mathbf{E}(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))^\top\right)\end{aligned} \tag{17}$$

for some $M \in \mathbb{R}^+$. On the other hand,

$$\begin{aligned}I(\mathbf{x}_0;\mathbf{e}^t) &\overset{(a)}{\geq} I(\mathbf{x}_u(0);\mathbf{e}^t) \\ &\overset{(b)}{\geq} I(\mathbf{x}_u(0);\hat{\mathbf{x}}_u(t)) \\ &\overset{(c)}{=} h(\mathbf{x}_u(0)) - h(\mathbf{x}_u(0)|\hat{\mathbf{x}}_u(t)) \\ &\overset{(d)}{=} h(\mathbf{x}_u(0)) - h(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t)|\hat{\mathbf{x}}_u(t)) \\ &\overset{(e)}{\geq} h(\mathbf{x}_u(0)) - h(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t)) \\ &\overset{(f)}{\geq} h(\mathbf{x}_u(0)) - \log(2\pi e)^n \\ &- \frac{1}{2}\log\left(\det \mathbf{E}\left[(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))^\top\right]\right)\,.\end{aligned} \tag{18}$$

Here, (a) follows from (P3) since $\mathbf{x}_u$ is a function of $\mathbf{x}$; (b) follows from (P3) since $\hat{\mathbf{x}}_u$ is a function of $\mathbf{e}^t$; (c) follows from (P1); (d) follows from (P4); (e) follows from (P1) and (f) is from (P6).

In what follows, we use the bound of $\log\left(\det \mathbf{E}(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))(\mathbf{x}_u(0)+\hat{\mathbf{x}}_u(t))^\top\right)$ in (17), and substitute it into (18) to obtain

$$\begin{aligned}\frac{I(\mathbf{x}_0;\mathbf{e}^t)}{t} \geq \frac{h(\mathbf{x}_u(0))}{t} - \frac{n\log(2\pi e)}{2t} \\ - \frac{M}{2t} + \log\left(\det(\exp(A_u))\right) .\end{aligned} \tag{19}$$

Note that

$$\log\left(\det(\exp(A_u))\right) = \sum_i \lambda_i(A_u) = \sum_i \Re(\lambda_i(A))^+ , \tag{20}$$

and taking the limit on both sides of (19), as $t \to \infty$, we obtain (14). ■

The following Lemma is a consequence of closed-loop causality. It will be used in subsequent derivations.

Lemma 3.2: Consider the feedback loop in Fig. 1, with all signals sampled with the given δ interval, $0 < \delta \leq \epsilon$. The following identity holds:

$$I(\mathbf{d}^{(\delta)}(i);[\mathbf{u}^{(\delta)}]^i,\mathbf{x}_0|[\mathbf{d}^{(\delta)}]^{i-1}) = 0\,, \quad \forall\; i \geq 1. \tag{21}$$

Proof:

$$\begin{aligned}I(\mathbf{d}^{(\delta)}(i);[\mathbf{u}^{(\delta)}]^i,\mathbf{x}_0|[\mathbf{d}^{(\delta)}]^{i-1}) &\overset{(a)}{\leq} I(\mathbf{d}^{(\delta)}(i);\mathbf{u}^{\delta i},\mathbf{u}^{(\delta)}(i),\mathbf{x}_0|[\mathbf{d}^{(\delta)}]^{i-1}) \\ &\overset{(b)}{\leq} I(\mathbf{d}^{(\delta)}(i);\mathbf{y}^{\delta i-\epsilon},\mathbf{n}^{\delta i}|[\mathbf{d}^{(\delta)}]^{i-1}) \\ &\overset{(c)}{\leq} I(\mathbf{d}^{(\delta)}(i);\mathbf{d}^{\delta i-\epsilon},\mathbf{x}_0,\mathbf{n}^{\delta i}|[\mathbf{d}^{(\delta)}]^{i-1}) \\ &\overset{(d)}{=} I(\mathbf{d}^{(\delta)}(i);\mathbf{d}^{\delta i-\epsilon},\mathbf{x}_0,\mathbf{n}^{\delta i},[\mathbf{d}^{(\delta)}]^{i-1}) - I(\mathbf{d}^{(\delta)}(i);[\mathbf{d}^{(\delta)}]^{i-1}) \\ &\overset{(e)}{=} I(\mathbf{d}^{(\delta)}(i);\mathbf{d}^{\delta i-\epsilon},[\mathbf{d}^{(\delta)}]^{i-1}) - I(\mathbf{d}^{(\delta)}(i);[\mathbf{d}^{(\delta)}]^{i-1}) \\ &\overset{(f)}{=} I(\mathbf{d}^{(\delta)}(i);\mathbf{d}^{(\delta)}(i-1)) - I(\mathbf{d}^{(\delta)}(i);\mathbf{d}^{(\delta)}(i-1)) \\ &= 0\end{aligned} \tag{22}$$

Here, (a) follows from (P3), since $[\mathbf{u}^{(\delta)}]^i$ is a function of $(\mathbf{u}^{\delta i},\mathbf{u}(\delta i))$; (b) also follows from (P3), since $(\mathbf{u}^{\delta i},\mathbf{u}(\delta i))$ is a function of $\mathbf{y}^{\delta i-\epsilon}$ and $\mathbf{n}^{\delta i}$; (c) also follows from (P3), since $\mathbf{y}^{\delta i-\epsilon}$ is a function of $\mathbf{d}^{\delta i-\epsilon}$, $\mathbf{x}_0$ and $\mathbf{n}^{\delta i}$; (d) follows from (P2); (e) follows from the assumption that $\mathbf{n}$, $\mathbf{x}_0$ and $\mathbf{d}$ are mutually independent; (f) follows from Markov property of $\mathbf{d}$. ■

We are ready to state the main theorem regarding closed-loop causality.

Theorem 3.3: Consider the closed loop shown in Fig. 1. The following inequality holds:

$$I(\mathbf{e}^t;\mathbf{u}^t) \geq I(\mathbf{d}^t;\mathbf{u}^t) + I(\mathbf{x}_0;\mathbf{e}^t)\,, \quad \forall\ t \in \mathbb{R}^+. \tag{23}$$

Proof: Given $t > 0$, we take $k+1$ samples of each of the signals $\mathbf{e}$, $\mathbf{d}$ and $\mathbf{u}$ over $[0,t)$, by sampling the interval $\delta(k) > 0$ to get the discretized signals $\{\mathbf{e}^{(\delta(k))}(i) : 1 \leq i \leq k\}$, $\{\mathbf{d}^{(\delta(k))}(i) : 1 \leq i \leq k\}$ and $\{\mathbf{u}^{(\delta(k))}(i) : 1 \leq i \leq k\}$ respectively. Notice also that $(k+1)\delta(k) = t$.

We expand the following mutual information by Kolmogrov's formula (P4) for any $1 \leq i \leq k$:

$$\begin{aligned}
&-I(\mathbf{d}^{(\delta(k))}(i);\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i|[\mathbf{d}^{(\delta(k))}]^{i-1})\\
&\qquad= I(\mathbf{d}^{(\delta(k))}(i);[\mathbf{d}^{(\delta(k))}]^{i-1}) - I(\mathbf{d}^{(\delta(k))}(i);[\mathbf{d}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i)\\
&\qquad\overset{(a)}{=} h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{d}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i) - h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{d}^{(\delta(k))}]^{i-1})\\
&\qquad\overset{(b)}{=} h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{e}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i) - h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{d}^{(\delta(k))}]^{i-1})\\
&\qquad\overset{(c)}{=} h(\mathbf{e}^{(\delta(k))}(i)|[\mathbf{e}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i) - h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{d}^{(\delta(k))}]^{i-1})\\
&\qquad\overset{(d)}{=} h(\mathbf{e}^{(\delta(k))}(i)|[\mathbf{e}^{(\delta(k))}]^{i-1}) - I(\mathbf{x}_0;\mathbf{e}^{(\delta(k))}(i)|[\mathbf{e}^{(\delta(k))}]^{i-1})\\
&\qquad\qquad -I([\mathbf{u}^{(\delta(k))}]^i;\mathbf{e}^{(\delta(k))}(i)|[\mathbf{e}^{(\delta(k))}]^{i-1},\mathbf{x}_0) - h(\mathbf{d}^{(\delta(k))}(i)|[\mathbf{d}^{(\delta(k))}]^{i-1})\,,
\end{aligned} \tag{24}$$

where (a) follows from (P1), (b) from the fact that $[\mathbf{e}^{(\delta(k))}]^{i-1} = [\mathbf{d}^{(\delta(k))}]^{i-1} + [\mathbf{u}^{(\delta(k))}]^{i-1}$ and therefore the map $([\mathbf{d}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i) \mapsto ([\mathbf{e}^{(\delta(k))}]^{i-1},\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i)$ is invertible, (c) from (P4) since $\mathbf{e}^{(\delta(k))}(i) = \mathbf{d}^{(\delta(k))}(i) + \mathbf{u}^{(\delta(k))}(i)$, and (d) is from (P4).

On the other hand, Lemma 3.2 claims that

$$I(\mathbf{d}^{(\delta(k))}(i);\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i|[\mathbf{d}^{(\delta(k))}]^{i-1}) = 0 \tag{25}$$

Summing up $-I(\mathbf{d}^{(\delta(k))}(i);\mathbf{x}_0,[\mathbf{u}^{(\delta(k))}]^i|[\mathbf{d}^{(\delta(k))}]^{i-1})$ from 1 to k, $\forall\ k \geq 1$, and considering (24), we have

$$
\begin{aligned}
0 &= \sum_{i=1}^{k} I(\mathbf{d}^{(\delta(k))}(i); \mathbf{x}_0, [\mathbf{u}^{(\delta(k))}]^i | [\mathbf{d}^{(\delta(k))}]^{i-1}) \\
&\overset{(a)}{=} h([\mathbf{e}^{(\delta(k))}]^k) - I(\mathbf{x}_0; [\mathbf{e}^{(\delta(k))}]^k) - h([\mathbf{d}^{(\delta(k))}]^k) \\
&- \sum_{i=1}^{k} I([\mathbf{u}^{(\delta(k))}]^i; \mathbf{e}^{(\delta(k))}(i) | [\mathbf{e}^{(\delta(k))}]^{i-1}, \mathbf{x}_0) \\
&\overset{(b)}{=} h([\mathbf{e}^{(\delta(k))}]^k) - h([\mathbf{e}^{(\delta(k))}]^k | [\mathbf{u}^{(\delta(k))}]^k + h([\mathbf{d}^{(\delta(k))}]^k | [\mathbf{u}^{(\delta(k))}]^k) \\
&- I(\mathbf{x}_0; [\mathbf{e}^{(\delta(k))}]^k) - h([\mathbf{d}^{(\delta(k))}]^k) \\
&\quad - \sum_{i=1}^{k} I([\mathbf{u}^{(\delta(k))}]^i; \mathbf{e}^{(\delta(k))}(i) | [\mathbf{e}^{(\delta(k))}]^{i-1}, \mathbf{x}_0) \qquad (26) \\
&\overset{(c)}{=} I([\mathbf{e}^{(\delta(k))}]^k; [\mathbf{u}^{(\delta(k))}]^k) - I(\mathbf{x}_0; [\mathbf{e}^{(\delta(k))}]^k) \\
&- \sum_{i=1}^{k} I([\mathbf{u}^{(\delta(k))}]^i; \mathbf{e}^{(\delta(k))}(i) | [\mathbf{e}^{(\delta(k))}]^{i-1}, \mathbf{x}_0) \\
&- I([\mathbf{d}^{(\delta(k))}]^k; [\mathbf{u}^{(\delta(k))}]^k) \\
&\overset{(d)}{\leq} I([\mathbf{e}^{(\delta(k))}]^k; [\mathbf{u}^{(\delta(k))}]^k) - I(\mathbf{x}_0; [\mathbf{e}^{(\delta(k))}]^k) \\
&- I([\mathbf{d}^{(\delta(k))}]^k; [\mathbf{u}^{(\delta(k))}]^k)
\end{aligned}
$$

Here (a) follows from (P5), (b) follows from (P4) since $h([\mathbf{e}^{(\delta(k))}]^k | [\mathbf{u}^{(\delta(k))}]^k) = h([\mathbf{d}^{(\delta(k))}]^k | [\mathbf{u}^{(\delta(k))}]^k)$, (c) follows from (P1) and (d) follows from the non-negativeness of mutual information.

Taking the limit as $k \to \infty$, we have $\delta(k) \to 0$, which consequently implies that

$$0 \leq I(\mathbf{e}^t; \mathbf{u}^t) - I(\mathbf{d}^t; \mathbf{u}^t) - I(\mathbf{x}_0; \mathbf{e}^t)\,. \qquad (27)$$

The inequality in (23) follows. ■

Remark 3.4: The quantity $\sum_{i=1}^{k} I([\mathbf{u}^{(\delta(k))}]^i; \mathbf{e}^{(\delta(k))}(i) | [\mathbf{e}^{(\delta(k))}]^{i-1}, \mathbf{x}_0)$ in the equation (b) of (26) has been defined in [35] as *directed information* from $[\mathbf{u}^{(\delta(k))}]^k$ to $[\mathbf{e}^{(\delta(k))}]^k$ conditioned by $\mathbf{x}_0$, and is denoted as $I([\mathbf{u}^{(\delta(k))}]^k \to [\mathbf{e}^{(\delta(k))}]^k | \mathbf{x}_0)$. One can define the continuous-time version of directed information by letting $k \to \infty$. A preliminary exploration of continuous-time directed information and its relation with optimal estimation theory has been reported recently in [36].

Remark 3.5: The proof of the above theorem is a direct consequence of the structure of the feedback and the causality of the plant $\mathcal{P}$, even though it is assumed that $\mathcal{P}$ is LTI (a rather specific case among all the complicated causal dynamics). Therefore, the generic relationship in the above theorem may be subject to possible extensions to various (more involved) plant dynamics. This potential is partially manifested in the recent development for switched systems [37]. In the subsequent discussion, however, we concentrate on and take advantage of the LTI nature of the plant.

An inequality for information rate is readily obtained by dividing both sides of (23) by t and letting t go to infinity (assuming that the limit exists). It is summarized in the following corollary.

Corollary 3.6: Given the closed-loop system in Fig. 1, we have

$$\bar{I}(\mathbf{e};\mathbf{u}) - \bar{I}(\mathbf{d};\mathbf{u}) \geq \bar{I}(\mathbf{x}_0;\mathbf{e}) \tag{28}$$

The subsequent theorem incorporates the mean square stability of the closed loop with the information rate inequality (28). Some stationarity assumptions are further enforced to derive a Bode-like formula. The details are summarized in the following theorem.

Theorem 3.7 (Bode-Like Formula): Assume the closed-loop system shown in Fig. 1 is mean-square stable. Then

$$\bar{I}(\mathbf{e};\mathbf{u}) \geq \bar{I}(\mathbf{d};\mathbf{u}) + \sum_i \Re(\lambda_i(A))^+ \,. \tag{29}$$

Furthermore, if $(\mathbf{d},\mathbf{u})$ and $(\mathbf{u},\mathbf{e})$ form stationary processes and $f_{\mathbf{u}} \in \mathbb{F}$ and $\mathbf{d}$ is a stationary Gaussian process, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega \geq \sum_i \Re(\lambda_i(A))^+ \,. \tag{30}$$

Proof: The inequality in (29) directly follows from (14) and (28). To obtain (30), first we have

$$\begin{aligned}
& I(\mathbf{e}^t;\mathbf{u}^t) - I(\mathbf{d}^t;\mathbf{u}^t) \\
& \overset{(a)}{=} \lim_{k\to\infty} \{ I([\mathbf{e}^{(\delta(k))}]^k;[\mathbf{u}^{(\delta(k))}]^k) - I([\mathbf{d}^{(\delta(k))}]^k;[\mathbf{u}^{(\delta(k))}]^k) \} \\
& \overset{(b)}{=} \lim_{k\to\infty} \{ h([\mathbf{e}^{(\delta(k))}]^k) - h([\mathbf{d}^{(\delta(k))}]^k) \} \\
& \overset{(c)}{\leq} \lim_{k\ \to\infty} \{ h([\bar{\mathbf{e}}^{(\delta(k))}]^k) - h([\mathbf{d}^{(\delta(k))}]^k) \} \\
& \overset{(d)}{=} \lim_{k\to\infty} \{ I([\bar{\mathbf{e}}^{(\delta(k))}]^k;[\bar{\mathbf{u}}^{(\delta(k))}]^k) - I([\mathbf{d}^{(\delta(k))}]^k;[\bar{\mathbf{u}}^{(\delta(k))}]^k) \} \\
& \overset{(e)}{=} I(\bar{\mathbf{e}}^t;\bar{\mathbf{u}}^t) - I(\mathbf{d}^t;\bar{\mathbf{u}}^t)\,,
\end{aligned} \tag{31}$$

where $(\bar{\mathbf{e}}, \bar{\mathbf{u}})$ stands for the Gaussian stationary process with the same covariance as $(\mathbf{e}, \mathbf{u})$. Here (a) follows from Lemma 2.6; (b) follows from (P1); (c) follows from (P6); (d) follows from (P1), and we use the fact that $h([\bar{\mathbf{e}}^{(\delta(k))}]^k | [\bar{\mathbf{u}}^{(\delta(k))}]^k) = h([\bar{\mathbf{d}}^{(\delta(k))}]^k | [\bar{\mathbf{u}}^{(\delta(k))}]^k), \forall k \in \mathbb{N}^+$; (e) follows from Lemma 2.6. Then it is straightforward to show that

$$\bar{I}(\mathbf{e};\mathbf{u}) - \bar{I}(\mathbf{d};\mathbf{u}) \leq \bar{I}(\bar{\mathbf{e}};\bar{\mathbf{u}}) - \bar{I}(\mathbf{d};\bar{\mathbf{u}}) \tag{32}$$

Since $f_{\mathbf{u}} \in \mathbb{F}$, Lemma 2.10 implies

$$\begin{aligned} &\bar{I}(\bar{\mathbf{e}};\bar{\mathbf{u}}) - \bar{I}(\mathbf{d};\bar{\mathbf{u}}) \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} \log\left(1 - \frac{f_{\mathbf{eu}}(\omega) f_{\mathbf{ue}}(\omega)}{f_{\mathbf{e}}(\omega) f_{\mathbf{u}}(\omega)}\right) d\omega + \frac{1}{4\pi}\int_{-\infty}^{\infty} \log\left(1 - \frac{f_{\mathbf{du}}(\omega) f_{\mathbf{ud}}(\omega)}{f_{\mathbf{d}}(\omega) f_{\mathbf{u}}(\omega)}\right) d\omega \\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} \log\left(\frac{f_{\mathbf{e}}(\omega)}{f_{\mathbf{d}}(\omega)} \cdot \frac{f_{\mathbf{d}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{du}}(\omega) f_{\mathbf{ud}}(\omega)}{f_{\mathbf{e}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{eu}}(\omega) f_{\mathbf{ue}}(\omega)}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right) d\omega\,. \end{aligned} \tag{33}$$

Here we have used the fact

$$\frac{f_{\mathbf{d}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{du}}(\omega) f_{\mathbf{ud}}(\omega)}{f_{\mathbf{e}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{eu}}(\omega) f_{\mathbf{ue}}(\omega)} = 1.$$

Indeed, since $\mathbf{d} = \mathbf{e} + \mathbf{u}$, then

$$\begin{aligned} f_{\mathbf{d}}(\omega) &= \int_0^{\infty} e^{-i\tau\omega} R_{\mathbf{e}+\mathbf{u}}(\tau) d\tau \\ &= \int_0^{\infty} e^{-i\tau\omega} (R_{\mathbf{e}}(\tau) + R_{\mathbf{e},\mathbf{u}}(\tau) + R_{\mathbf{u},\mathbf{e}}(-\tau) + R_{\mathbf{u}}(\tau)) d\tau \\ &= f_{\mathbf{e}}(\omega) + f_{\mathbf{eu}}(\omega) + f_{\mathbf{ue}}(\omega) + f_{\mathbf{u}}(\omega)\,, \end{aligned} \tag{34}$$

and

$$\begin{aligned} f_{\mathbf{du}} &= \int_0^{\infty} e^{-i\tau\omega} R_{\mathbf{e}+\mathbf{u},\mathbf{u}}(\tau) d\tau \\ &= \int_0^{\infty} e^{-i\tau\omega} (R_{\mathbf{e},\mathbf{u}} + R_{\mathbf{u}})(\tau) d\tau \\ &= f_{\mathbf{eu}}(\omega) + f_{\mathbf{u}}(\omega)\,. \end{aligned} \tag{35}$$

Hence, (34) and (35) give

$$\begin{aligned} &\frac{f_{\mathbf{d}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{du}}(\omega) f_{\mathbf{ud}}(\omega)}{f_{\mathbf{e}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{eu}}(\omega) f_{\mathbf{ue}}(\omega)} \\ &= \frac{(f_{\mathbf{e}}(\omega) + f_{\mathbf{eu}}(\omega) + f_{\mathbf{ue}}(\omega) + f_{\mathbf{u}}(\omega)) f_{\mathbf{u}}(\omega) - (f_{\mathbf{u}}(\omega) + f_{\mathbf{eu}}(\omega))(f_{\mathbf{u}}(\omega) + f_{\mathbf{ue}}(\omega))}{f_{\mathbf{e}}(\omega) f_{\mathbf{u}}(\omega) - f_{\mathbf{eu}}(\omega) f_{\mathbf{ue}}(\omega)} \\ &= 1\,. \end{aligned}$$

The proof is complete. ■

Remark 3.8: The equation (30) is formally identical to the inequality version of Bode's integral developed in the classical case [14], where a time delay is introduced to force the residual of $\log|S(s)|$ vanish at infinity for strictly proper plants. The same type of time delay in the course of our derivation is introduced to ensure closed-loop causality, so that the sequential relations among the signals residing in Fig. 1 are revealed by using information theoretical machineries.

Remark 3.9: For derivation of Bode's integral formula (30) from the information conservation law in (28) we have hinged on *stationary* signals for simplicity. Nonetheless, similar argument can be also extended to *asymptotically stationary* cases with minor modifications.

IV. Negative Component of Bode's Integral

In the section, we investigate the lower bound of $\bar{I}(\mathbf{d};\mathbf{u})$, with additional assumptions that $\mathbf{d}$ and $\mathbf{e}$ are mutually wide sense stationary and $\mathbf{d}$ is Gaussian. As shown in the subsequent result, the lower bound of $\bar{I}(\mathbf{d};\mathbf{u})$ is shown to be given by the negative portion of Bode's integral.

The following theorem summarizes the main result.

Theorem 4.1: Consider the feedback closed loop given in Fig. 1, where $\mathbf{d}$ and $\mathbf{e}$ are mutually wide-sense stationary and $\mathbf{d}$ is a Gaussian process. If $f_{\mathbf{u}}(\omega)$ is bounded away from zero, then the following inequality holds

$$\bar{I}(\mathbf{d};\mathbf{u}) \geq -\frac{1}{2\pi}\int_{-\infty}^{\infty} (\log S_{\mathbf{d},\mathbf{e}}(\omega))^{-}\, d\omega \tag{36}$$

Proof: To begin with, we consider the following Wiener predictor

$$L(j\omega) = \frac{f_{\mathbf{d},\mathbf{u}}(\omega)}{f_{\mathbf{u}}(\omega)} e^{j\omega\epsilon}\,,$$

which represents the minimal mean square error prediction of $\mathbf{d}$, given the observation of the entire time history of $\mathbf{u}$ with the time delay ϵ. To obtain a causal prediction of $\mathbf{d}(t)$ by using the possibly noncausal $L(j\omega)$, we define the following predictor:

$$\hat{\mathbf{d}}(t) = L(s)\lfloor \mathbf{u}(t)\rfloor_t\,,$$

where $\lfloor\cdot\rfloor_t$ stands for the truncation operator.

The above Wiener predictor is now used to lower bound the quantity $\bar{I}(\mathbf{u};\mathbf{d})$. First, the process $\mathbf{d}(\tau), 0 \leq \tau < t$ is sampled with interval $\delta(k) = \frac{t}{k+1}$, leading to

$$
\begin{aligned}
I([\mathbf{d}^{(\delta(k))}]^k;\mathbf{u}^{t-\epsilon}) &\overset{(a)}{\geq} I([\mathbf{d}^{(\delta(k))}]^k;\hat{\mathbf{d}}^t)\\
&\overset{(b)}{\geq} I([\mathbf{d}^{(\delta(k))}]^k;[\hat{\mathbf{d}}^{(\delta(k))}]^k)\\
&\overset{(c)}{=} h([\mathbf{d}^{(\delta(k))}]^k)-h([\mathbf{d}^{(\delta(k))}]^k|[\hat{\mathbf{d}}^{(\delta(k))}]^k)\\
&\overset{(d)}{\geq} h([\mathbf{d}^{(\delta(k))}]^k)-h([\tilde{\mathbf{d}}^{(\delta(k))}]^k)\\
&\overset{(e)}{=} h([\mathbf{d}^{(\delta(k))}]^k)-h([\mathbf{d}^{(\delta(k))}]^k|[\hat{\mathbf{d}}^{(\delta(k))}]^k)+h([\tilde{\mathbf{d}}^{(\delta(k))}]^k|[\hat{\mathbf{d}}^{(\delta(k))}]^k)-h([\tilde{\mathbf{d}}^{(\delta(k))}]^k)\\
&= I([\mathbf{d}^{(\delta(k))}]^k;[\hat{\mathbf{d}}^{(\delta(k))}]^k)-I([\tilde{\mathbf{d}}^{(\delta(k))}]^k;[\hat{\mathbf{d}}^{(\delta(k))}]^k)\,,
\end{aligned}
$$

where $\tilde{\mathbf{d}} := \mathbf{d} - \hat{\mathbf{d}}$. Here (a) follows from (P3), since $\hat{\mathbf{d}}^t$ is a function of $\mathbf{u}^{t-\epsilon}$; (b) follows from (P3), since $[\hat{\mathbf{d}}^{(\delta(k))}]^k$ is a function $\hat{\mathbf{d}}^t$; (c) follows from (P1); (d) follows from the fact that conditioning reduces entropy; (e) follows from $h([\mathbf{d}^{(\delta(k))}]^k|[\hat{\mathbf{d}}^{(\delta(k))}]^k) = h([\tilde{\mathbf{d}}^{(\delta(k))}]^k|[\hat{\mathbf{d}}^{(\delta(k))}]^k)$.

By applying Lemma 2.6, we have

$$
I(\mathbf{d}^t;\mathbf{u}^{t-\epsilon}) \geq I(\mathbf{d}^t;\hat{\mathbf{d}}^t) - I(\tilde{\mathbf{d}}^t;\hat{\mathbf{d}}^t)\,,
$$

which in turn gives the limiting case

$$
\bar{I}(\mathbf{d};\mathbf{u}) \geq \bar{I}(\mathbf{d};\hat{\mathbf{d}}) - \bar{I}(\tilde{\mathbf{d}};\hat{\mathbf{d}})\,. \tag{37}
$$

Note that $\mathbf{d}$ and $\hat{\mathbf{d}}$ are Gaussian and stationarily correlated and $f_{\mathbf{d}} \in \mathbb{F}$, and from Lemma 2.6 we have

$$
\begin{aligned}
&\bar{I}(\mathbf{d};\hat{\mathbf{d}}) - \bar{I}(\tilde{\mathbf{d}};\hat{\mathbf{d}})\\
&= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{f_{\mathbf{d}\hat{\mathbf{d}}}(\omega)f_{\hat{\mathbf{d}}\mathbf{d}}(\omega)}{f_{\mathbf{d}}(\omega)f_{\hat{\mathbf{d}}}(\omega)}\right)d\omega + \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{f_{\tilde{\mathbf{d}}\hat{\mathbf{d}}}(\omega)f_{\hat{\mathbf{d}}\tilde{\mathbf{d}}}(\omega)}{f_{\tilde{\mathbf{d}}}(\omega)f_{\hat{\mathbf{d}}}(\omega)}\right)d\omega\\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(\frac{f_{\mathbf{d}}(\omega)}{f_{\tilde{\mathbf{d}}}(\omega)}\right)d\omega\\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(\frac{f_{\mathbf{d}}(\omega)}{f_{\mathbf{d}}(\omega)-|L(j\omega)|^2 f_{\mathbf{u}}(\omega)}\right)d\omega\,,
\end{aligned}
$$

where we have used the fact

$$
f_{\tilde{\mathbf{d}}}(\omega) = f_{\tilde{\mathbf{d}}}(\omega) = f_{\mathbf{d}}(\omega) - |L(j\omega)|^2 f_{\mathbf{u}}(\omega).
$$

We then note that

$$
|L(j\omega)| = \frac{|f_{\mathbf{du}}(\omega)|}{|f_{\mathbf{u}}(\omega)|} \geq \frac{\Re(f_{\mathbf{du}}(\omega))}{f_{\mathbf{u}}(\omega)} = \frac{f_{\mathbf{du}}(\omega)+f_{\mathbf{ud}}(\omega)}{2f_{\mathbf{u}}(\omega)} = \frac{f_{\mathbf{e}}(\omega)-f_{\mathbf{d}}(\omega)-f_{\mathbf{u}}(\omega)}{2f_{\mathbf{u}}(\omega)}
$$

Therefore (37) is further written as

$$\bar{I}(\mathbf{d},\mathbf{u}) \geq \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(\frac{4f_{\mathbf{d}}(\omega)f_{\mathbf{u}}(\omega)}{-f_{\mathbf{d}}^2(\omega)-f_{\mathbf{e}}^2(\omega)-f_{\mathbf{u}}^2(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{e}}(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{u}}(\omega)+2f_{\mathbf{u}}(\omega)f_{\mathbf{e}}(\omega)}\right)d\omega \tag{38}$$

Taking the maximum value of the right hand side of (38), by calculating the extreme value point of the quadratic function $-f_{\mathbf{d}}^2(\omega)-f_{\mathbf{e}}^2(\omega)-f_{\mathbf{u}}^2(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{e}}(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{u}}(\omega)+2f_{\mathbf{u}}(\omega)f_{\mathbf{e}}(\omega)$, under the constraint that $f_u(\omega) > 0$, we have

$$\sup_{f_u(\omega)>0}\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(\frac{4f_{\mathbf{d}}(\omega)f_{\mathbf{u}}(\omega)}{-f_{\mathbf{d}}^2(\omega)-f_{\mathbf{e}}^2(\omega)-f_{\mathbf{u}}^2(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{e}}(\omega)+2f_{\mathbf{d}}(\omega)f_{\mathbf{u}}(\omega)+2f_{\mathbf{u}}(\omega)f_{\mathbf{e}}(\omega)}\right)d\omega$$
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}(\log S_{\mathbf{d},\mathbf{e}}(\omega))^{-}\,d\omega\,.$$

The relation in (36) follows from the fact that (38) holds also for all $f_{\mathbf{u}}(\omega) > 0$.

■

Once the inequality (36) is obtained, we can employ the inequality (44) later in Section VI to obtain the following theorem.

Theorem 4.2: Consider the closed loop shown in Fig. 1, where $\mathbf{e}$ and $\mathbf{d}$ are assumed jointly stationary, with $\mathbf{d}$ being a Gaussian process. If the closed loop is mean square stable, then the following upper bound holds:

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}(\log S_{\mathbf{d},\mathbf{e}}(\omega))^{-}\,d\omega \leq \bar{I}((\mathbf{x}(0),\mathbf{d});\mathbf{u}) - \sum_i \Re(\lambda_i(A))^{+}\,, \tag{39}$$

Remark 4.3: The upcoming discussion in Section VI will show that $\bar{I}((\mathbf{x}(0),\mathbf{d});\mathbf{u})$ represents the total information flow in the closed loop. Therefore, the inequality in (36) implies that the negative portion of the Bode integral (where $S_{\mathbf{d},\mathbf{e}}(\omega) < 1$) is determined by both the degree of open-loop instability and the information rate transmitted through it. It can be observed from (36) that if $\bar{I}((\mathbf{x}(0),\mathbf{d});\mathbf{u}) = \sum_i \Re(\lambda_i(A))^{+}$, then the $S_{\mathbf{d},\mathbf{e}}(\omega) \geq 1$ for all ω. Moreover, the same observation shows that, to achieve a desirable shaping of the sensitivity function, one needs a larger information transmission rate to allow for a less constraint on the negative part of $\log S_{\mathbf{d},\mathbf{e}}(\omega)$.

V. Achievable Lower Bound of Bode's Integral for LTI Systems

This section is devoted to further investigation of the tightness of the resulting Bode's integral. As it has been shown in (30), the sum of the unstable poles serves as a lower bound on the

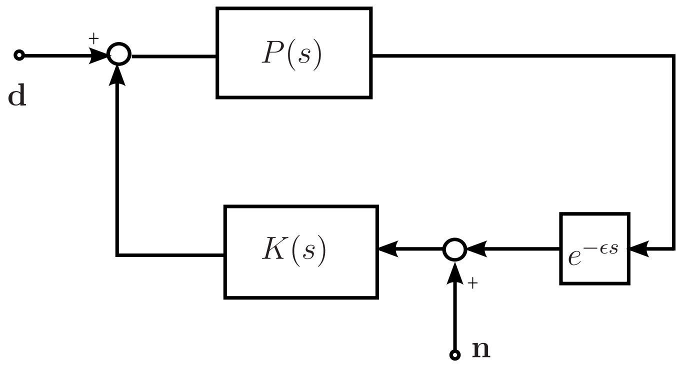

Fig. 2. Linear Stochastic Closed Loop

log-integral of the sensitivity function; however, the conservativeness of this *inequality* remains unclear. One can intuitively conclude that the controller noise $\mathbf{n}$ contributes to the increase of $\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(S_{\mathbf{d},\mathbf{e}}(\omega)\right)d\omega$ by making $\mathbf{e}$ noisier within some frequency range. Detailed analysis of this issue is given subsequently, where the controller and the plant are given by LTI systems.

We now specialize the problem to the closed-loop configuration, shown in Fig. 2, where $P(s)$ is strictly proper and minimum phase, and the unstable poles are denoted as $\{p_1, p_2, ..., p_N\}$. In addition, we choose a proper stable stabilizing controller $K(s)$. The controller noise $\mathbf{n}(t)$ is a stationary (possibly colored) Gaussian process with zero mean; the disturbance signal $\mathbf{d}$ is a stationary Gaussian process. A candidate $\mathbf{d}$ can be expressed as the following Itô integral, also known as Ornstein-Uhlenbeck Brownian motion:

$$\mathbf{d}(t) = b\int_0^t e^{-a(t-u)}dW_u\,,$$

where $a > 0$ and $b \neq 0$ are real numbers, W_t is a standard Wiener process. The initial conditions for both $P(s)$ and $K(s)$ are set to 0.

Note that the closed loop is stable (with sufficiently small $\epsilon > 0$), and that $\mathbf{d}$ and $\mathbf{n}$ are independent. We have

$$f_{\mathbf{e}}(\omega) = \frac{f_d(\omega)}{|1-P(j\omega)K(j\omega)e^{-j\omega\epsilon}|^2} + \frac{|K(j\omega)|^2 f_{\mathbf{n}}(\omega)}{|1-P(j\omega)K(j\omega)e^{-j\omega\epsilon}|^2}\,.$$

Subsequently, the sensitivity function is obtained as

$$S_{\mathbf{d},\mathbf{e}}(\omega) = \sqrt{\frac{f_{\mathbf{e}}(\omega)}{f_{\mathbf{d}}(\omega)}} = \frac{\sqrt{1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}}}{|1-P(j\omega)K(j\omega)e^{-j\omega\epsilon}|} \tag{40}$$

Next, we prove the following theorem regarding the log-integral of sensitivity.

Theorem 5.1: Consider the closed loop shown in Fig 2. The following equality holds

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\log S_{\mathbf{d},\mathbf{e}}(\omega)d\omega = \sum_i \Re(\lambda_i(A))^+ + \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega\,. \tag{41}$$

Proof: By using (40), we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\log S_{\mathbf{d},\mathbf{e}}(\omega)d\omega =& \frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(\frac{1}{|1-P(j\omega)K(j\omega)e^{-j\omega\epsilon}|}\right)d\omega + \\ & \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega\,.\end{aligned}$$

Notice that $1/(1-K(s)P(s))$ is stable and proper. Then we employ the same argument as in the proof of Theorem 3.1.4 in [14] to obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(\frac{1}{|1-P(j\omega)K(j\omega)e^{-j\omega\epsilon}|}\right)d\omega \\ &= \frac{1}{2\pi j}\oint_{\mathscr{C}}\log\left(\frac{1}{|1-P(s)K(s)e^{-s\epsilon}|}\right)ds \\ &= p_1+\ldots+p_N = \sum_i \Re(\lambda_i(A))^+\,.\end{aligned}$$

Here $\mathscr{C}$ denotes the right half plane closed contour, which has a sufficiently large radius and circumvents all the unstable poles of $P(s)$ [14]. The same integration can also be calculated by a simplified methodology developed in [38]. The proof is complete. ■

The positive term $\kappa := \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega$ in (41) presents an additional performance limitation, on the top of the sum of the unstable poles. In order to gain some insight, we now illustrate the significance of this term from different perspectives.

- Although it is not easy to quantify κ in general (yet a special case is given later in Lemma 5.2 towards explicit calculation of κ), we can roughly estimate its value by observing the magnitudes of $f_{\mathbf{d}}(\omega)$, $f_{\mathbf{n}}(\omega)$ and $K(j\omega)$. It becomes evident that, both a lower noise-to-disturbance ratio $f_{\mathbf{n}}(\omega)/f_{\mathbf{d}}(\omega)$ and a smaller controller magnitude $|K(j\omega)|$ lead to a less restrictive limitation on the closed loop.
- From information theoretical point of view, the expression of κ reminds of the mutual information rate of a continuous-time additive Gaussian channel [26]. For the non-feedback additive Gaussian channel shown in Fig. 3, the input/ouput mutual information can be calculated by Lemma 2.10 as

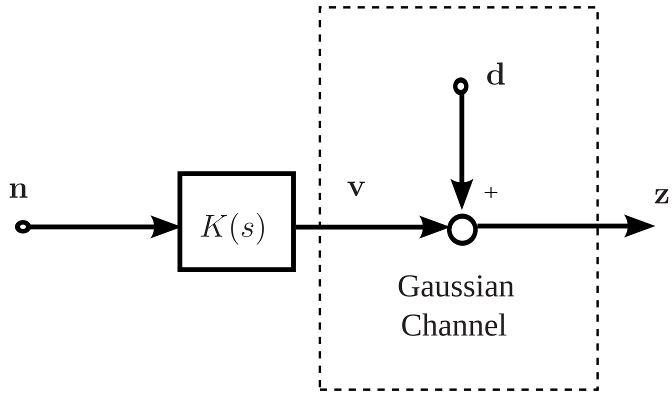

Fig. 3. Additive Gaussian channel

$$\begin{aligned}\bar{I}(\mathbf{v};\mathbf{z}) &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|f_{\mathbf{vz}}(\omega)|^2}{f_{\mathbf{v}}(\omega)f_{\mathbf{z}}(\omega)}\right)d\omega\\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1-\frac{|f_{\mathbf{v}}(\omega)|^2}{f_{\mathbf{v}}(\omega)(f_{\mathbf{v}}(\omega)+f_{\mathbf{d}}(\omega))}\right)d\omega\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega = \kappa\,.\end{aligned}$$

The above interpretation of κ shows that the extra amount of performance limitation is induced by the mutual information rate between the propagated controller noise $\mathbf{v}$ and the observation $\mathbf{z}$. To reduce the mutual information rate, one can reduce the uncertainty of the channel source $\mathbf{v}$, which can be done by either lowering the magnitude of $K(s)$, or denoising the controller noise $\mathbf{n}$. We also note that the power constraint in the channel can be satisfied by a proper choice of $K(s)$.

- κ can be also related to the H_∞ entropy [39]. Suppose there exits a proper transfer function $M(s)$ such that

$$\frac{1}{2}\frac{|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}}{1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}} = M(-j\omega)M(j\omega).$$

Then the above relation leads to

$$\kappa = \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega = -\frac{\gamma^2}{2\pi}\int_{-\infty}^{\infty}\log\left(1-\gamma^{-2}M(-j\omega)M(j\omega)\right)d\omega\,,$$

which is exactly the expression of H_∞ entropy of $M(s)$ with disturbance rejection level $\gamma = 1/\sqrt{2}$. It has been shown that the minimal H_∞ entropy controller (i.e. central controller) is equivalent to a suboptimal H_∞ controller ($\|M\|_{H_\infty} \leq \gamma$) [39]. Therefore the above observation actually proposes a way to minimize κ by resorting to various H_∞ methodologies for the design of $K(s)$. While the detailed development along this direction is not given here, the readers are encouraged to look into this interesting problem as it provides a potential link between H_∞ theory and information theory.

Next we will show that, under some mild assumptions, κ can be obtained explicitly, where we assume that $f_{\mathbf{d}}(\omega)$ and $f_{\mathbf{n}}(\omega)$ are rational and admit the following spectral factorizations:

$$f_{\mathbf{d}}(\omega) = \phi_{\mathbf{d}}(-j\omega)\phi_{\mathbf{d}}(j\omega), \qquad f_{\mathbf{n}}(\omega) = \phi_{\mathbf{n}}(-j\omega)\phi_{\mathbf{n}}(j\omega)\,.$$

Lemma 5.2: Assume that $K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ admits a minimal realization $(A_k, b_k, c_k^\top, d_k)$ with A_k being Hurwitz. Moreover, assume that there exists a matrix $Q > 0$ solving the following algebraic Riccati equation (ARE):

$$A_k^\top Q + QA_k - \frac{1}{1+d_k^2}Qb_kb_k^\top Q + \frac{1}{1+d_k^2}c_kc_k^\top = 0\,. \tag{42}$$

Then

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega = \frac{1}{\sqrt{1+d_k^2}}b_k^\top Qb_k + \frac{d_k}{\sqrt{1+d_k^2}}c_k^\top b_k$$

Proof: We will first obtain the following spectral factorization:

$$1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)} = H(-j\omega)H(j\omega)\,,$$

where $H(s) = -\frac{1}{\sqrt{1+d_k^2}}(b_k^\top Q + d_kc_k^\top)(s\mathbb{I}-A_k)^{-1}b_k + \sqrt{1+d_k^2}$. Indeed, it can be verified that

$$\begin{aligned}
&H(-s)H(s)\\
&= \left(\frac{1}{\sqrt{1+d_k^2}}b_k^\top(s\mathbb{I}+A_k^\top)^{-1}(Qb_k+d_kc_k)+\sqrt{1+d_k^2}\right)\times\\
&\qquad\left(\frac{-1}{\sqrt{1+d_k^2}}(b_k^\top Q+d_kc_k^\top)(s\mathbb{I}-A_k)^{-1}b_k+\sqrt{1+d_k^2}\right)\\
&= b_k^\top(-s\mathbb{I}-A_k^\top)^{-1}c_kc_k^\top(s\mathbb{I}-A_k^\top)^{-1}b_k + d_kc_k^\top(s\mathbb{I}-A_k^\top)^{-1}b_k + d_kc_k^\top(-s\mathbb{I}-A_k^\top)^{-1}b_k + 1 + d_k^2\\
&= K(-s)\frac{\phi_{\mathbf{n}}(-s)}{\phi_{\mathbf{d}}(-s)}K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}+1\,.
\end{aligned}$$

Next, note that both $K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ and $1/K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ are analytic on the right half plane. Hence, we have

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(1+|K(j\omega)|^2\frac{f_{\mathbf{n}}(\omega)}{f_{\mathbf{d}}(\omega)}\right)d\omega \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(H(-j\omega)H(j\omega)\right)d\omega \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\log\left(|H(-j\omega)|^2\right)d\omega \\
&= \Re\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\log\left(H(-j\omega)\right)d\omega\right) \\
&= \Re\left(\frac{1}{2\pi j}\oint_{\mathscr{D}}\log\left(H(-s)\right)ds\right),
\end{aligned}
$$

where $\mathscr{D}$ denotes a contour encompassing from $-j\infty$ to $j\infty$ and enclosing $\mathbb{C}^+$. The value of the integration along the contour can then be evaluated by using the residue of $\log\left(H(-s)\right)$ about $s=\infty$, which is calculated as

$$
Res(\log\left(H(-s)\right);\infty) = -\lim_{s\to\infty}s(H(-s)-H(\infty)) = \frac{1}{\sqrt{1+d_k^2}}b_k^\top Qb_k + \frac{d_k}{\sqrt{1+d_k^2}}c_k^\top b_k.
$$

Residue theorem in turn yields

$$
\Re\left(\frac{1}{2\pi j}\oint_{\mathscr{D}}\log\left(H(-s)\right)ds\right) = \frac{1}{\sqrt{1+d_k^2}}b_k^\top Qb_k + \frac{d_k}{\sqrt{1+d_k^2}}c_k^\top b_k\,.
$$

The proof is complete. ∎

In summary, the following theorem holds.

Theorem 5.3: Consider the closed loop shown in Fig. 3, and assume that $K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ admits a minimal realization $(A_k, b_k, c_k^\top, d_k)$ and A_k is Hurwitz, and $Q>0$ is the unique solution to the ARE in (42). Then

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\log S_{\mathbf{d},\mathbf{e}}(\omega)d\omega = \sum_i \Re(\lambda_i(A))^+ + \frac{1}{\sqrt{1+d_k^2}}b_k^\top Qb_k + \frac{d_k}{\sqrt{1+d_k^2}}c_k^\top b_k\,. \tag{43}
$$

Remark 5.4: The condition that $K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ needs to be proper does not impose a significant restriction on the class of closed loops, for which we can do the same calculations as in Theorem 5.3; one can always choose stabilizing $K(s)$ with higher relative degree, rendering $K(s)\frac{\phi_{\mathbf{n}}(s)}{\phi_{\mathbf{d}}(s)}$ proper.

VI. Information Rate Inequality & Control with Communication Constraints

Another information rate inequality regarding the closed-loop stability, based on the framework in Section III, is obtained in this section. By using it, we investigate the stabilization problem, where the communication channel is modeled as a continuous-time Gaussian channel with certain Signal-to-Noise Ratio (SNR) level constraint.

The following lemma provides a lower bound for the mutual information rate $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$, which accounts for total information rate flow in the loop. Further insight into $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$ is provided later in Remark 6.2.

Lemma 6.1: Consider the closed-loop system shown in Fig. 1. We have the following inequality:

$$\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u}) \geq \bar{I}(\mathbf{x}_0; \mathbf{e}) + \bar{I}(\mathbf{d}; \mathbf{u})\,. \tag{44}$$

Proof: Using Kolmogorov's formula (P2), we have

$$I((\mathbf{x}_0, \mathbf{d}^t); \mathbf{u}^t) = I(\mathbf{x}_0; \mathbf{u}^t|\mathbf{d}^t) + I(\mathbf{u}^t; \mathbf{d}^t)\,, \tag{45}$$

where $t \in \mathbb{R}^+$ is arbitrary time instance. We can lower bound $I((\mathbf{x}_0, \mathbf{d}^t); \mathbf{u}^t)$ as

$$\begin{aligned} & I((\mathbf{x}_0, \mathbf{d}^t); \mathbf{u}^t) \\ & \overset{(a)}{=} I(\mathbf{x}_0; \mathbf{e}^t|\mathbf{d}^t) + I(\mathbf{u}^t; \mathbf{d}^t) \\ & \overset{(b)}{=} I(\mathbf{x}_0; \mathbf{e}^t) - I(\mathbf{x}_0; \mathbf{d}^t) + I(\mathbf{x}_0; \mathbf{d}^t|\mathbf{e}^t) + I(\mathbf{u}^t; \mathbf{d}^t) \\ & \overset{(c)}{=} I(\mathbf{x}_0; \mathbf{e}^t) + I(\mathbf{x}_0; \mathbf{d}^t|\mathbf{e}^t) + I(\mathbf{u}^t; \mathbf{d}^t) \\ & \overset{(d)}{\geq} I(\mathbf{x}_0; \mathbf{e}^t) + I(\mathbf{u}^t; \mathbf{d}^t)\,. \end{aligned} \tag{46}$$

Here (a) follows from the fact that $I(\mathbf{x}_0; \mathbf{u}^t|\mathbf{d}^t) = I(\mathbf{x}_0; \mathbf{u}^t + \mathbf{d}^t|\mathbf{d}^t) = I(\mathbf{x}_0; \mathbf{e}^t|\mathbf{d}^t)$; (b) follows from (P2); (c) follows from the independence of $\mathbf{d}$ and $\mathbf{x}_0$; and (d) follows from the fact that $I(\mathbf{x}_0; \mathbf{d}^t|\mathbf{e}^t) \geq 0$. We have obtained the following inequality:

$$I((\mathbf{x}_0, \mathbf{d}^t); \mathbf{u}^t) \geq I(\mathbf{x}_0; \mathbf{e}^t) + I(\mathbf{u}^t; \mathbf{d}^t)\,. \tag{47}$$

The conclusion is readily obtained by dividing the terms on both sides of (47) by t and taking the limit as $t \to \infty$. ■

Remark 6.2: To illustrate the importance of $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$, we consider the block diagrams shown in Fig. 4, which recast the closed loop in Fig. 1 into a typical analog communication scheme with feedback [27]. The "message" to be transmitted is composed of the two independent sources $\mathbf{x}_0$ and $\mathbf{d}(t)$, and $\mathbf{u}(t)$ is the channel output. We can also identify the "transmitter" and the "channel" in this "communication system" accordingly, though in our current setup they do not function the same way as their names suggest. It turns out to be clear that $\bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u})$ represents the input/ouput information rate, and therefore Lemma 6.1 indicates that the total information flow of the closed loop is bounded from below by the contributions of the initial value and the disturbance.

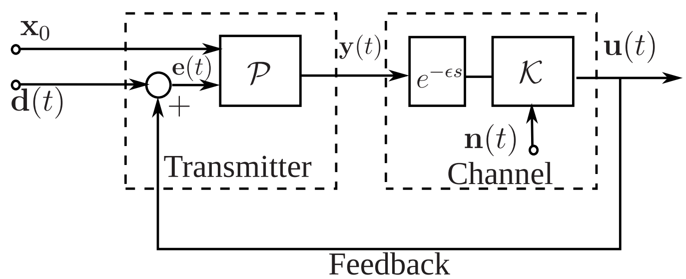

Fig. 4. Closed loop configuration from the communication perspective

We can then define the *feedback capacity* of the closed loop in Fig. 1 as

$$\mathcal{C}_f := \sup_{\mathbf{x}_0, \mathbf{d}} \bar{I}((\mathbf{x}_0, \mathbf{d}); \mathbf{u}) .$$

Notice that the discrete-time and non-causal version of the feedback capacity has been introduced in several papers, as [40] and [3].

Remark 6.3: Feedback capacity does not depend solely on the type of the communication channel in the closed loop. It is in fact determined by the structure of the overall system: the plant, the controller and the communication scheme. Similar to what happens to almost all the quantities in information theory, calculating the exact value of the feedback capacity is technically challenging.

To take the closed-loop stability into consideration, we further elaborate the inequality (44) to get the following theorem.

Theorem 6.4: If the closed-loop system shown in Fig. 1 with feedback capacity $\mathcal{C}_f$ is mean-square stable, then

$$\bar{I}(\mathbf{u}; \mathbf{d}) \le \mathcal{C}_f - \sum_i \Re(\lambda_i(A))^+ . \tag{48}$$

Example: Stabilization with Gaussian Channel Constraint

Next we focus on the continuous time additive white Gaussian noise (AWGN) channel with input power constraint. This particular type of a communication channel, rooted in Shannon's celebrated work [12], has been intensively studied for its theoretical and practical significance in various papers, [13] [41] and [42]. To consider the Gaussian channel in a feedback loop, we adopt the same scheme as in [22], which is shown in Fig. 5. Here, $\mathcal{P}$ is the same LTI system as in (11) and $\mathbf{y}(t) = \mathbf{x}(t)$; $K \in \mathbb{R}^{1\times n}$ is the control gain matrix; $\mathbf{u}(t)$ is the channel input with power constraint $\mathbf{E}[\mathbf{u}^2(t)] \leq \mathscr{P}, \;\; \forall\, t \geq 0$, for some power level $\mathscr{P} > 0$; $\mathbf{d}(t)$ is a Gaussian white noise process with SDF $f_{\mathbf{d}} \equiv \Phi > 0$.

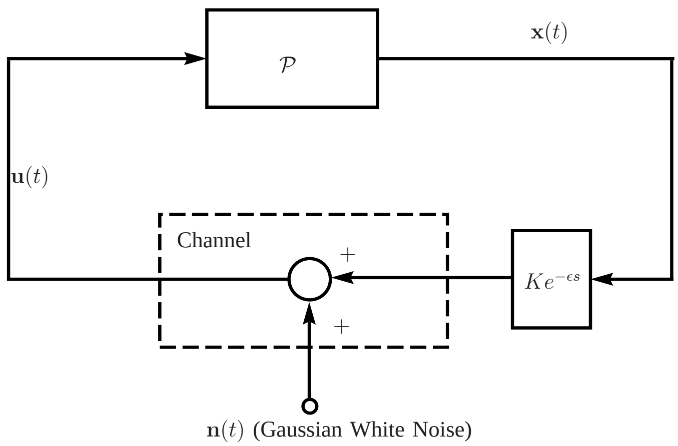

Fig. 5. Feedback control in the presence of a Gaussian channel

The channel capacity $\mathcal{C}$ can be obtained by the following formula [27]:

$$\mathcal{C} = \frac{\mathscr{P}}{2\Phi}\,. \tag{49}$$

Regarding the closed-loop system stability, we have the following theorem.

Theorem 6.5: If the closed-loop system shown in Fig. 5 is mean-square stable, then the following relationship holds:

$$\frac{\mathscr{P}}{2\Phi} \geq \sum_i \Re(\lambda_i(A))^+\,. \tag{50}$$

Proof: Note that $\mathbf{d} \equiv 0$ and the fact that feedback does not change the capacity of memoryless white Gaussian additive channels implies

$$\frac{\mathscr{P}}{2\Phi} = \mathcal{C} = \mathcal{C}_f = \sup_{\mathbf{x}_0} \bar{I}(\mathbf{x}_0; \mathbf{u})\,.$$

Therefore (48) is reduced to

$$\frac{\mathscr{P}}{2\Phi} = \mathcal{C} \geq \bar{I}(\mathbf{x}_0; \mathbf{u}) \geq \sum_i \Re(\lambda_i(A))^+ . \tag{51}$$

The proof is complete. ■

Remark 6.6: This result provides a sufficient condition to solve *Problem 1* in [22]. A similar condition is also obtained in [21], where the authors have used the result from [41] on mutual information rate of a Gaussian channel. Different from [22], the method used here is purely information theory-based, and may be applied to more general systems rather than LTI.

VII. Conclusion

In this paper we derived the continuous-time information conservation laws in a causal closed loop feedback setting as an extension from the well established discrete-time case. For the purpose of this extension, we resort to mutual information rate rather than differential entropy rate, whose behavior is not desirable in the continuous-time setting. As a result of the aforementioned conservation laws, a Bode-type integral formula is obtained, for which we have used mutual information integral inequalities instead of the widely used Kolmogorov's formula. We also pursue an in-depth investigation into the resulting Bode integral in terms of its tightness and its relation with communication constraints. These conservation laws have also shown the ability of handling particular problems such as control with limited information.

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