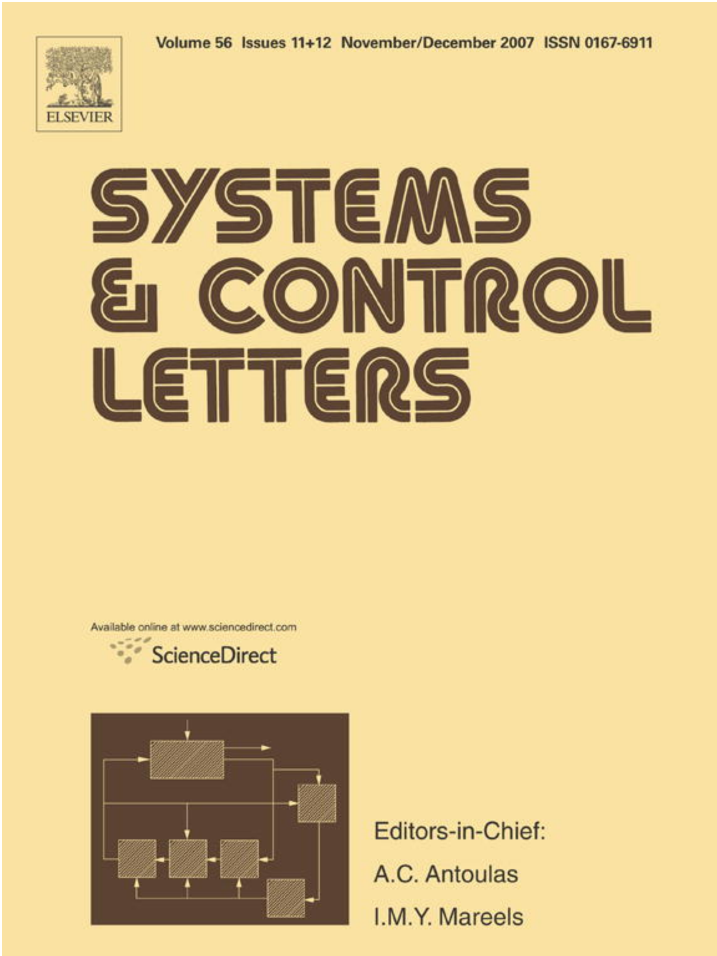
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On initial value and terminal value problems for Hamilton–Jacobi equation☆

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Abstract

First order partial differential equations (PDE) are often the main tool to model problems in optimal control, differential games, image processing, physics, etc. Dependent upon the particular application, the boundary conditions are specified either at the initial time instant, leading to an initial value problem (IVP), or at the terminal time instant, leading to a terminal value problem (TVP). The IVP and TVP have in general different solutions. Thus introducing a new model in terms of a first order PDE one has to consider both possibilities of IVP and TVP, unless there is a direct physical indication. In this paper we also particularly answer the following question: how should the initial value at the initial surface and the terminal value at the terminal surface be coordinated in order to generate the same solution? One may expect that for a given initial value the consistent terminal value is the value of the IVP solution at the terminal surface. The second (time-varying) example in this paper shows that, generally, this is not true for *non-smooth* initial conditions. We discuss also the difference between the IVP and TVP formulations, the connection between the Hamiltonians arising in IVP and TVP.

Keywords: Hamilton–Jacobi equation; Viscosity solutions; Initial value problem; Terminal value problem

1. Introduction

For nonlinear first-order PDEs one has a boundary value problem of the form [10,7]:

$$F(x, u, \partial u/\partial x) = 0, \quad x \in \Omega \subset R^n, \tag{1}$$

$$u(x) = v(x), \quad x \in M \subset \partial\Omega.$$

This form also captures the Hamilton–Jacobi equation

$$\frac{\partial u}{\partial t} + H\left(x, \frac{\partial u}{\partial x}, t\right) = 0 \quad (x, t) \in \Omega, \tag{2}$$

if we treat the time variable as one of the components of the state vector x: $(x, t) \to x$.

If $u(x)$, $F(x, u, p) \in C^2$, then the local construction of the solution to the problem (1) is known to be reduced to the integration of the following system of (regular) characteristics ($p = \partial u/\partial x$) [7,1]:

$$\dot{x} = F_p, \quad \dot{u} = \langle p, F_p \rangle, \quad \dot{p} = -F_x - pF_u. \tag{3}$$

In optimal control and differential games, the structure of the Hamiltonians F and H is related to the dynamic equations and the cost function of the problem (see Section 3). In the problems of image processing, and particularly shape-from-shading, the structure of the Hamiltonians changes due to the physics of the problem. The 2D image is described by the *intensity function* $I(x)$, where $x = (x_1, x_2) \in G$ is a point of the image region G. Under certain assumptions about the reflection physics, the intensity of the image appears to be a function of the form: $I(x) = \langle \gamma, n(x) \rangle$, where $n(x)$ is the unit normal to the surface at the point $x \in G$, and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the fixed direction from which the light is coming. Expressing the normal through

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the partials of the surface height $x_3 = u(x_1, x_2)$ leads to the following first-order PDE [12,4,15]:

$$I(x) = \frac{-\gamma_1 u_{x_1} - \gamma_2 u_{x_2} + \gamma_3}{\sqrt{1 + u_{x_1}^2 + u_{x_2}^2}}. \tag{4}$$

In many problems of physics and control one or both of the functions $u(x)$, $F(x, u, p)$ may be non-smooth and the solution to the problem (1) must be understood in a generalized sense. One of the most powerful approaches to the generalization of the solution developed during the recent decades is the theory of viscosity solutions [11,3]. As one can see from the definition in the next section the viscosity theory assigns to Eqs. (1), (2), (4) two types of solutions—that for the IVP and TVP. This paper is aimed to give a comparison of the solutions to IVP and TVP and to show that when introducing a new model in terms of a first order PDE one has to analyze both possibilities of IVP and TVP.

The method of characteristics, appropriately generalized in [13], is one of the attractive construction methods for the generalized (viscosity) solutions [8,9].

This paper is organized as follows: in Section 2, we review the basics from the viscosity theory and the related boundary conditions. In Section 3, we formulate the initial and the TVPs in optimal control. Section 4 presents some illustrative observations. Conclusions are summarized in Section 5.

2. Viscosity solution and boundary conditions

To get a unique solution to the PDE (4), one needs to specify appropriate boundary (initial, terminal) conditions [10,5]. A comprehensive analysis of possible boundary conditions in shape-from-shading problems is given in [12]. Further, one needs to distinguish between two types of problems—the initial value problem (IVP) and the TVP. The surface M in (1) for the IVP and the TVP correspondingly has the form:

$$M = M_0 = \{(x, t) \in R^{n+1} : t = t_0\},$$
$$M = M_1 = \{(x, t) \in R^{n+1} : t = t_1\}, \tag{5}$$

so that M is a "part" of the overall boundary $\partial\Omega = M_0 \cup M_1$. In time-invariant setting the surface M may coincide with $\partial\Omega$, or may need to be specified specially. The viscosity solution theory suggests general approach for specifying M and formulating the boundary conditions [3]. In differential game theory, M is called the "usable" part of the boundary, and certain necessary conditions for M are given in [10].

The boundary value problem formulation in the shape-from-shading problems is quite different from optimal control. In the shape-from-shading problem one needs to know the shape on a part of the boundary. Such a boundary shape generally is not available. So, one can reconstruct the shape up to an arbitrary boundary function. On the other hand, the shape reconstruction is sufficient up to an additive constant: $u(x) - C$. When the boundary surface degenerates into a point this constant C can be considered as the value of the solution at that point. Such boundary conditions are known and used in differential games [10] in the framework of the method of characteristics.

In several papers on shape-from-shading the extremum points (maximum or minimum) of the solution are characterized as a degenerate boundary. The boundary conditions may have a special type of singularity. As shown in [12], generally three types of edges (parts of boundary) may arise: apparent contours, grazing light edges and shadow edges. The first and second types of edges are shown to have a singularity similar to that in the state constraint optimal control problem [2,6]. The method of (generalized) characteristics states that a family of regular or singular characteristics is projected on this boundary [14].

Finally, we recall the definition of a viscosity solution for IVP and TVP in terms of scalar test functions $\varphi(x) \in C^1(\Omega)$ [3].

Definition. A continuous function $u : \Omega + M \to R^1$ is called the viscosity solution of IVP (1) if:

(1) $u(x) = v(x),\ x \in M$;
(2) for every test function $\varphi(x) \in C^1(\Omega)$ such that local minimum (maximum) of $u(x) - \varphi(x)$ is attained at $x^0 \in \Omega$ the following inequality holds:

$$F(x^0, u(x^0), \varphi_x(x^0)) \geqslant 0,$$
$$(F(x^0, u(x^0), \varphi_x(x^0)) \leqslant 0). \tag{6}$$

For the definition of a viscosity solution to a TVP the inequalities in (6) must be changed to opposite ones with, generally, a different terminal surface M.

3. Initial and terminal value problems in optimal control

TVPs typically arise in optimal control, while IVPs more often model problems in physics or mechanics. Though many practical problems of calculus of variations, optimal control or differential games generate a TVP, for some constructions IVP may also be considered. A TVP arises when one is interested in the cost function as a function of the left end of the optimal path, while an IVP arises in case of the interest in the right end of the path.

In this section we use the letter u for control variable, while the scalar solutions to the PDE will be denoted as S or V.

3.1. Time-varying optimal control and calculus of variations' problem

Consider a traditional formulation of the optimal control problem, which includes the relevant ODE:

$$\dot{x} = f(x, u, t), \quad u \in U \subset R^m, \quad t \in [t_0, t_1], \tag{7}$$

the endpoint conditions:

$$x(t_0) = x^0, \quad x(t_1) = x^1, \quad x \in R^n, \tag{8}$$

and the cost function:

$$J[x,u]=\int_{t_0}^{t_1} L(x,u,t)\,dt+\Phi_0(x(t_0),t_0)+\Phi_1(x(t_1),t_1)\to \min_{u(t)}. \quad (9)$$

Here x is the n-dimensional state vector, u is the m-dimensional vector of the control inputs, U is a convex constraint set in R^m, and $t_1>t_0$.

Introduce the function on both (left and right) ends of the optimal path:

$$J^*(x^0,t_0;x^1,t_1) = \min_{u(t)}\left(\int_{t_0}^{t_1} L(x,u,t)\,dt+\Phi_0(x^0,t_0)+\Phi_1(x^1,t_1)\right).$$

The dynamic programming approach gives that the function (of the left end)

$$S(x,t)=\min_{(x^1,t_1)} J^*(x,t;x^1,t_1),\quad (x^1,t_1)\in M_1,$$

is the solution of the (terminal value) problem

$$\frac{\partial S}{\partial t}+H^l\left(x,\frac{\partial S}{\partial x},t\right)=0,\quad (x,t)\in\Omega, \quad (10)$$

$$S(x,t)=\Phi_1(x,t),\quad (x,t)\in M_1\ \ (M_1\subset\partial\Omega),$$

while the right end function

$$V(x,t)=\min_{(x^0,t_0)} J^*(x^0,t_0;x,t),\quad (x^0,t_0)\in M_0,$$

is the solution of the (initial value) problem

$$\frac{\partial V}{\partial t}+H^r\left(x,\frac{\partial V}{\partial x},t\right)=0,\quad (x,t)\in\Omega, \quad (11)$$

$$V(x,t)=\Phi_0(x,t),\quad (x,t)\in M_0\quad (M_0\subset\partial\Omega),$$

where M_1 and M_0 are the given terminal and initial surfaces (manifolds).

Eqs. (10) and (11) are written in terms of different Hamiltonians, which we call left (H^l) and right (H^r) Hamiltonians. Dynamic programming gives the following relations for them:

$$H^l(x,p,t)=\min_{u\in U}(\langle p,f(x,u,t)\rangle+L(x,u,t)),$$
$$H^r(x,p,t)=\max_{u\in U}(\langle p,f(x,u,t)\rangle-L(x,u,t)), \quad (12)$$

so that one has

$$H^r(x,p,t)=-H^l(x,-p,t). \quad (13)$$

Similar relation holds for the extended Hamiltonians, i.e. including the time partials $\partial S/\partial t$, $\partial V/\partial t$. In a typical optimal control problem the left Hamiltonian in (12) is used.

For the calculus of variations problem, where $\dot x=f\equiv u$, $U=R^n$, the Hamiltonians H^l and H^r are the Legandre transforms of $L(x,u,t)=L(x,\dot x,t)$:

$$H^l(x,p,t)=\min_{\dot x}(\langle p,\dot x\rangle+L(x,\dot x,t))\quad (p=-L_{\dot x}(x,\dot x,t)),$$
$$H^r(x,p,t)=\max_{\dot x}(\langle p,\dot x\rangle-L(x,\dot x,t))\quad (p=L_{\dot x}(x,\dot x,t)), \quad (14)$$

and the Hamilton–Jacobi equations have the same form (10), (11).

3.2. Time invariant optimal control and calculus of variations' problem

Let the functions f, L, Φ_0, Φ_1 in (11)–(13) be independent of time variable, $f(x,u)$, $L(x,u)$, $\Phi_0(x)$, $\Phi_1(x)$, and the sets Ω, M_0, M_1 be subsets of R^n. Then the (value or Bellman) functions $S(x)$, $V(x)$ are also time-invariant and satisfy the dynamic programming equations:

$$H^l(x,p)=\min_{u\in U}(\langle p,f(x,u)\rangle+L(x,u))=0\quad (p=\partial S/\partial x),$$
$$H^r(x,p)=\max_{u\in U}(\langle p,f(x,u)\rangle-L(x,u))=0\quad (p=\partial V/\partial x), \quad (15)$$

subject to appropriate boundary conditions on M_1, M_0, see (10), (11). Comparing with (10), (11), there are no time partials in Eqs. (15).

In case of the calculus of variations problem, the above equations take the form:

$$H^l(x,p)=\min_{\dot x}(\langle p,\dot x\rangle+L(x,\dot x))=0\quad (p=\partial S/\partial x),$$
$$H^r(x,p)=\max_{\dot x}(\langle p,\dot x\rangle-L(x,\dot x))=0\quad (p=\partial V/\partial x). \quad (16)$$

Note that for the time-optimal control problem, when $L(x,u)\equiv 1$, usually a more simple Hamiltonian is considered so that the dynamic programming equation (e.g. for the "right" case) takes the form

$$H^r(x,p)=1\quad (H^r(x,p)=\max_{u\in U}\langle p,f(x,u)\rangle).$$

It is interesting to note that the Hamilton–Jacobi equation of a similar form arises in a calculus of variations' problem with the Lagrangian homogeneous in $\dot x$ (see the next subsection).

Remark 1. The above Hamilton–Jacobi equations in calculus of variations require the assumption

$$\det L_{\dot x\dot x}\neq 0$$

in a certain set of variables $(x,\dot x,t)$. For the time-invariant case this means that the Lagrangian $L(x,\dot x)$ is non-homogeneous in $\dot x$ (see the next subsection). Such inequality is required for the existence of a solution for $\dot x$ to equations $p=\pm L_{\dot x}(x,\dot x,t)$ (extremum conditions in (14) and (16)) via the implicit function theorem. Dependent upon the set U and the other parameters of the problem this condition may be crucial for optimal control problems as well.

3.3. Calculus of variations problem with a homogeneous Lagrangian

Consider a calculus of variations problem with a homogeneous Lagrangian:

$$L(x, \lambda\dot{x}) = \lambda L(x, \dot{x}), \quad \lambda > 0.$$

Differentiating this identity in λ with respect to λ and letting $\lambda = 1$ one can get the following representation for $L(x, \dot{x})$:

$$L(x, \dot{x}) = \langle \dot{x}, L_{\dot{x}}(x, \dot{x})\rangle.$$

This means that the Hamiltonians (16) identically vanish: $H^l(x, p) \equiv 0$, $H^r(x, p) \equiv 0$, since one has $p = \pm L_{\dot{x}}(x, \dot{x})$. Furthermore, differentiating the latter identity in $\dot{x}$ with respect to $\dot{x}$, one gets the other identity: $L_{\dot{x}\dot{x}}\dot{x} = 0$. Since this linear equation in $\dot{x}$ has non-trivial solutions one has $\det L_{\dot{x}\dot{x}} = 0$.

The above considerations show that for the functions $S(x)$, $V(x)$ one needs the Hamilton–Jacobi equation in a different form than the one in (16). The Hamilton–Jacobi theory for the problems with homogeneous Lagrangians one can find, for instance, in [16]. Though the degeneracy $\det L_{\dot{x}\dot{x}} = 0$ does not allow to solve the equation $p = L_{\dot{x}}(x, \dot{x}, t)$ for $\dot{x}$, the other condition,

$$\det \tfrac{1}{2}(L^2)_{\dot{x}\dot{x}} = \det(L_{\dot{x}}L_{\dot{x}}^{\mathrm{T}} + L\,L_{\dot{x}\dot{x}}) \neq 0,$$

may be fulfilled in generic case. Here $L_{\dot{x}}$ is a column-vector, $L_{\dot{x}}^{\mathrm{T}}$ denotes the transpose of $L_{\dot{x}}$. The above condition allows to solve the equation

$$p = (L^2)_{\dot{x}}/2 = L\,L_{\dot{x}},$$

thus defining a function $\dot{x} = \omega(x, p)$. One can show that this function is also homogeneous in p: $\omega(x, \lambda p) = \lambda\omega(x, p)$, $\lambda > 0$. Introduce now the (right) Hamiltonian as

$$H^r(x, p) = L(x, \omega(x, p)) \tag{17}$$

and recall that the first variation formula gives [7]

$$\partial S/\partial x = L_{\dot{x}}(x, \dot{x}), \quad \partial V/\partial x = -L_{\dot{x}}(x, \dot{x}). \tag{18}$$

Using now the vector $\partial S/\partial x = p/L(x, \dot{x})$ as the second argument in the Hamiltonian and assuming that $L > 0$ one can get

$$H^r\left(x, \frac{\partial S}{\partial x}\right) = L\left(x, \omega\left(x, \frac{p}{L}\right)\right) = \frac{L(x, \omega(x, p))}{L(x, \dot{x})} = 1. \tag{19}$$

Similarly, in accordance to (13), letting $H^l(x, p) = -L(x, \omega(x, -p))$, one can get

$$H^l\left(x, \frac{\partial V}{\partial x}\right) = -1. \tag{20}$$

Note that in many problems the degeneracy $\det L_{\dot{x}\dot{x}} = 0$ can be removed by choosing an appropriate component of the vector x as a new independent variable. Then Eqs. (19), (20) take the form (10), (11), (14).

One of the best and important illustrations for a homogeneous problem is the problem of the shortest (geodesic) line on a Riemannian manifold when

$$L(x, \dot{x}) = \sqrt{\langle G(x)\dot{x}, \dot{x}\rangle}, \quad (L^2(x, \dot{x}) = \langle G(x)\dot{x}, \dot{x}\rangle).$$

Here $G(x)$ is a symmetric non-singular matrix, the metric tensor of a Riemannian manifold with local coordinates x, the Lagrangian L is positive, homogeneous, and of order one. Further,

$$(L^2)_{\dot{x}\dot{x}}/2 = G(x), \quad p = G(x)\dot{x}, \quad \dot{x} = G^{-1}(x)p = \omega(x, p),$$

$$H^r(x, p) = -H^l(x, p) = \sqrt{\langle G^{-1}(x)p, p\rangle}.$$

Thus, both functions $S(x)$, $V(x)$ satisfy the same Hamilton–Jacobi equation:

$$\left\langle G^{-1}(x)\frac{\partial S}{\partial x}, \frac{\partial S}{\partial x}\right\rangle = 1.$$

It is proved in [16] that in general homogeneous case one has

$$L^2(x, \dot{x}) = \langle G(x, \dot{x})\dot{x}, \dot{x}\rangle, \quad G(x, \lambda\dot{x}) = G(x, \dot{x}), \quad \lambda > 0,$$

i.e. the matrix G now depends upon $\dot{x}$ and is homogeneous of the order zero in $\dot{x}$.

3.4. A note on terminology

The terms IVP and TVP are quite clear for time varying problems like (10), (11), in which the boundary values typically are specified on the surfaces

$$M_0 = \{(x, t) \in R^{n+1} : t = t_0\}, \quad M_1 = \{(x, t) \in R^{n+1} : t = t_1\}, \tag{21}$$

where M_0 corresponds to the initial time instant, and M_1 to the terminal time instant. The characteristic flow goes from M_0 to M_1.

The situation is not that obvious in case of time-invariant problems. The traditional terminology says that we have just a boundary value problem for any of the two possible definitions of the solution in (6). Dependent upon the signs of the inequalities in (6), the problem (1) produces two viscosity solutions. It is necessary to distinguish between these two solutions. To avoid the introduction of new terminology, it is natural to call these two definitions and corresponding viscosity solutions as IVP and TVP solutions. The IVP is the one, for which the signs in (6) correspond to the IVP in the time-varying problem, and the characteristic flow departs from the boundary surface $M = M_0$. For the TVP, the signs in (6) correspond to the TVP in the time-varying problem, while the characteristic flow approaches the boundary surface $M = M_1$. The direction of the flow is naturally defined using the auxiliary "time"-variable introduced by the equations of regular characteristics (3): $\dot{x} = \mathrm{d}x/\mathrm{d}t = F_p$.

Thus, the pairs (F, M_0) and (F, M_1), consisting of the Hamiltonian and the boundary surfaces, produce two different viscosity solutions in general.

In some problems the boundary surfaces for the IVP and TVP may coincide, $M_0 = M_1 = M$, say, $M = \partial\Omega$. This is not the case for the problems with boundary surfaces of the form (21). We will denote the viscosity solution $u(x)$ of the IVP (of the TVP) produced by the pair (F, M) by $I(F, M)$ (by $T(F, M)$). Generally, $I(F, M) \neq T(F, M)$. One can verify that $I(F, M) = T(-F, M)$. Thus, $I(F, M) \neq T(F, M) = I(-F, M)$.

In that sense, one can state that the equations $F = 0$ and $-F = 0$ have in general different viscosity solutions, which is not the case for the classical solution. If one prefers to solve IVP rather than TVP, then the inverse time must be used, which is equivalent to changing the sign in front of F (or H).

Generally, with a given Hamiltonian $H(x, p)$ one can associate eight boundary value problems since for each of the four Hamiltonians,

$$H(x, p), \quad -H(x, p), \quad -H(x, -p), \quad H(x, -p),$$

one can formulate a TVP or IVP. While this is mathematically well-defined and possible, only half of them are physically meaningful (see examples).

Thus, summarizing the considerations of IVP and TVP in terms of the Bellman function as a function of the left or the right end of an optimal path, one can state that the following Hamiltonians may arise in different problems:

$$\pm H(x, \pm p).$$

The sign in front of H switches between the IVP and TVP, while the sign in front of p switches between the left and the right end.

We consider here three examples demonstrating the diversity of the IVP and TVP solutions of the left and right HJB-equations.

4. Examples

4.1. Control of a car

This example illustrates the time-invariant case with different boundary surfaces M and different solutions for IVP and TVP. Consider the time-optimal control problem which is a particular case of the game of two cars by Isaacs [10]. The dynamics are given by

$$\dot{x} = -uy, \quad \dot{y} = ux - 1, \quad |u| \leqslant 1.$$

The control objective is to bring the state vector (x, y) to the terminal circle,

$$M: \quad x^2 + y^2 = l^2 \quad (0 \leqslant l < 1),$$

in minimum-time.

The corresponding Hamiltonian has the form:

$$H(x, y, p, q) = \min_u (-uyp + uxq - q) + 1 = -|qx - py| - q + 1.$$

The optimal time $V(x, y)$ for this problem is the TVP solution to

$$H(x, y, \partial V/\partial x, \partial V/\partial y) = 0, \quad V(x, y) = 0, \quad (x, y) \in M.$$

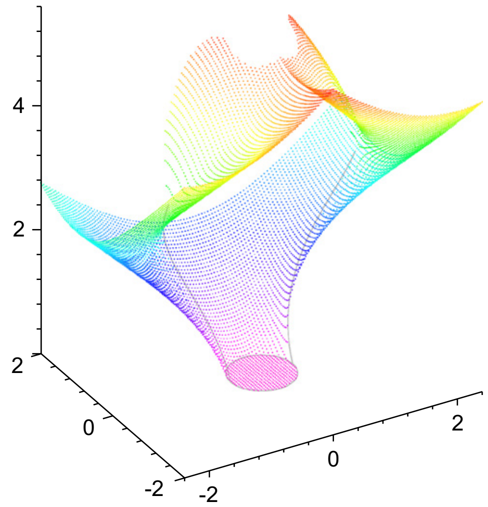

Fig. 1. The value function $V(x, y)$.

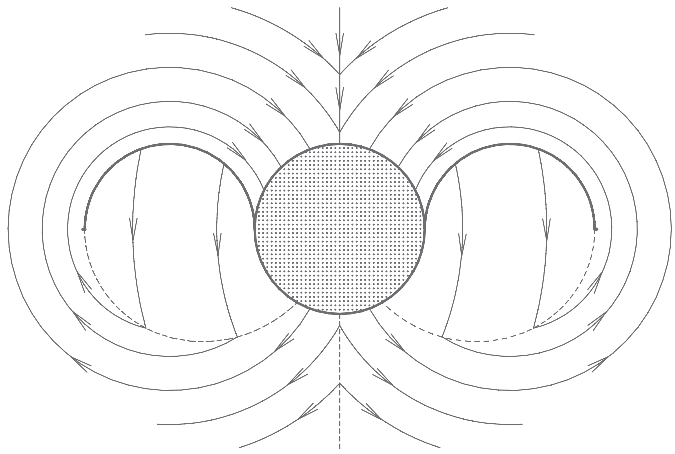

Fig. 2. The set of optimal paths (characteristics).

The graph of the function $V(x, y)$ is given in Fig. 1, and the corresponding optimal paths are shown in Fig. 2. This solution can be reconstructed from the considerations in Isaacs' book [10]. The so-called usable part of the boundary, the set M_1, happens to be the upper part of the circle M. Usually this problem is solved by switching to the IVP for the Hamiltonian with the opposite sign $-H(x, y, p, q)$ with the same usable boundary M_1.

The formally considered IVP for the problem:

$$H(x, y, \partial V/\partial x, \partial V/\partial y) = 0, \quad V(x, y) = 0, \quad (x, y) \in M_0,$$

has the usable part M_0 (lower part of M) and a solution of negative sign.

One can find physical interpretation only for the following four cases:

IVPs for:

$$\text{problem 2}: -H(x, y, p, q) = 0,$$
$$\text{problem 3}: -H(x, y, -p, -q) = 0.$$

TVPs for:

problem 1 : $H(x, y, p, q) = 0$,
problem 4 : $H(x, y, -p, -q) = 0$.

For the problems 1,2 one has $M = M_1$ and for the problems 3,4—$M = M_0$.

The solutions $V_i(x, y)$ of the ith problem are related as follows:

$$V_1(x, y) = V_2(x, y), \quad V_3(x, y) = V_4(x, y) = V_1(-x, -y).$$

The following remark explains the difference between the problems 1,2 and 3,4. The above dynamic equations $\dot{x} = -uy$, $\dot{y} = ux - 1$ are written in Cartesian coordinates (x, y) of the center of the terminal circle in the frame connected with the car. In the problems 1,2 with the usable part M_1 the value function depends on the left end.

In the problems 3,4 with the usable part M_0 one can consider the car's coordinates in the frame connected with the center of the terminal circle. The value function depends on the right end. Denote this coordinates as X, Y. One has $X=-x$, $Y=-y$, which maps M_1 onto M_0. Thus,

$$\dot{X} = -uY, \quad \dot{Y} = uX + 1.$$

This leads to the Hamiltonian of the problem 4.

4.2. A 2D differential game

By this example we investigate the consistency between the initial and terminal values, which may generate the same solution. Such a question may be addressed both to a time-invariant or time-varying problem, the latter being more convenient for a geometrical illustration. Consider a time-varying IVP with non-smooth initial data given by [13]:

$$F(x, y, p, q) = p + \sqrt{a^2 + q^2} - x\sqrt{b^2 + q^2} = 0, \quad x > 0, \tag{22}$$

$$u(0, y) = -|y| + cy \quad (p = \partial u/\partial x,\ q = \partial u/\partial y,\ a, b, c = \text{const}).$$

One can show that Eq. (22) is the HJBI equation for the following fixed-time differential game with one spatial variable and a non-smooth terminal cost function:

$$\dot{y} = u_1 + (T - t)v_1, \quad 0 \leqslant t \leqslant T, \quad u_1^2 + u_2^2 \leqslant 1, \quad v_1^2 + v_2^2 \leqslant 1,$$

$$J = -|y(T)| + cy(T) + \int_0^T (au_2 + b(T - t)v_2)\,dt \to \min_{u_i} \max_{v_i}.$$

One can see now that Eq. (22) is written in reverse time $x = T - t$.

IVP solution: Set $a = b$ and $c = 0$. We first describe the function providing the IVP solution and then give a sketch of its construction. Such solution is given by

$$u(x, y) = \min[u^+(x, y), u^-(x, y)] = -|y| + \sqrt{a^2+1}(x^2/2 - x),$$
$$u^\pm(x, y) = \mp y + \sqrt{a^2 + 1}(x^2/2 - x), \tag{23}$$

everywhere in the half-plane $y \geqslant 0$ except for the region:

$$x \geqslant 1, \quad |y| \leqslant \frac{(x-1)^2}{2\sqrt{a^2+1}},$$

where the solution is equal to the following smooth function $v(x, y)$:

$$v(x, y) = -\sqrt{a^2 + 1}/2 + a\sqrt{(x-1)^4/4 - y^2}.$$

The smooth branches $u^\pm(x, y)$ can be constructed using the system of (regular) characteristics with the corresponding initial conditions:

$$\dot{x} = F_p = 1, \quad \dot{y} = F_q = q(1 - x)/\sqrt{a^2 + q^2},$$

$$\dot{u} = pF_p + qF_q = p + qF_q, \quad \dot{p} = -F_x = \sqrt{a^2 + q^2},$$

$$\dot{q} = -F_y = 0,$$

$$x(0) = 0, \quad y(0) = s, \quad u = -|s|, \quad p(0) = -\sqrt{a^2 + 1},$$

$$q(0) = \operatorname{sign} s, \quad s \in R. \tag{24}$$

The first equation $\dot{x} = 1$ means that the variable x actually coincides with "time", the independent parameter of the differentiation in (24). Since q is a constant along the solutions of (24) and $\dot{y}$, $\dot{u}$ are linear in x the integration of the system (see [5, Chapter 3]) leads to the solutions (23) quadratic in x. The integration gives also that regular characteristics of the problem (24) form the following one-parametric family of parabolas in the (x, y)-plane with the vertical symmetry axes $x = 1$, see Fig. 3 (left):

$$y - C = \frac{qx(1 - x/2)}{\sqrt{a^2 + q^2}}, \quad C = \text{const}. \tag{25}$$

The solution $u = v(x, y)$ in the set between the two extremal parabolas with $C = \pm 1/2\sqrt{a^2 + 1}$ and $x > 1$ is constructed by integration of the equation for $\dot{u}$ along the following funnel of one-parametric family of characteristics (with the parameter q), starting at the point $(1, 0)$:

$$y(x) = -q(x-1)^2/[2\sqrt{a^2 + q^2}], \quad |q| \leqslant 1. \tag{26}$$

One can see that the resulting solution $u(x, y)$ is smooth (and satisfies the equation in classical sense) everywhere except for the segment $y = 0$, $0 \leqslant x \leqslant 1$, where $u^+(x, y) = u^-(x, y)$. The graph of this solution and the related characteristics are shown on Fig. 3 (left). One has to check that the viscosity requirements are fulfilled on that segment. First of all, one can show that there is no smooth test function $\varphi(x, y)$ such that $\min(u - \varphi)$ is attained at a some point of the considered segment. In [13], one parametric family of test functions is suggested

$$\varphi(x, y) = (1 + \theta)u^+(x, y)/2 + (1 - \theta)u^-(x, y)/2, \quad |\theta| \leqslant 1,$$

for which $\max(u - \varphi)$ is attained at the points of the segment, except for the right end $x = 1$. Thus one has to check the inequality

$$F(x, 0, \varphi_x, \varphi_y) = (x - 1)(\sqrt{a^2 + 1} - \sqrt{a^2 + \theta^2}) \leqslant 0,$$

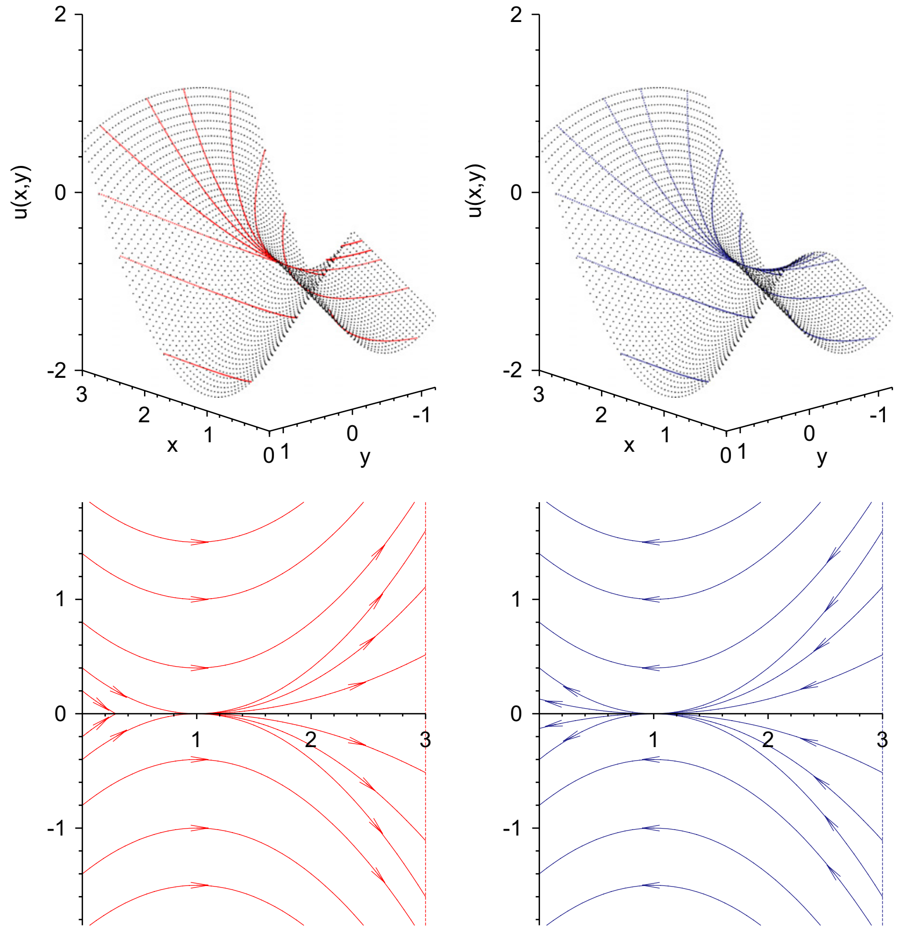

Fig. 3. Solution of IVP (left) and TVP (right) and related characteristics.

for each $|\theta| \leqslant 1$ and $0 \leqslant x < 1$. For the constructed function it is obviously satisfied, though for other values of the problem parameters it may be violated, see [13]. Consideration of the point $x=1$, $y=0$ proceeds as follows. First, for the constructed solution $u(x, y)$ the function $u(x, 0)$ is smooth in x at $x=1$ and $u_x(x, 0)=0$. This means that for every test function $\varphi(x, y)$, supplying the maximum (or minimum) to $u-\varphi$ at $x=1$, $y=0$, one has $\varphi_x=0$. Next, one observes that $F(1, 0, p, q)=p$, thus $F(1, 0, \varphi_x, \varphi_y)=\varphi_x=0$ regardless of the value of φ_y.

TVP solution: Fix now some positive value of x, say $x_T=3$. For this fixed x_T, the above IVP solution takes the values:

$$u(3, y) = -|y| + \frac{3}{2}\sqrt{a^2+1}, \quad |y| \geqslant \frac{2}{\sqrt{a^2+1}},$$
$$u(3, y) = -\sqrt{a^2+1}/2 + a\sqrt{4-y^2}, \quad |y| \leqslant \frac{2}{\sqrt{a^2+1}}. \qquad (27)$$

Now consider the TVP on the left half-plane $x \leqslant 3$ for Eq. (22) with the terminal conditions (27). One can show that the solution of this TVP coincides with the previous IVP solution everywhere except for the region

$$x \leqslant 1, \quad |y| \leqslant \frac{(x-1)^2}{2\sqrt{a^2+1}},$$

where the TVP solution equals $v(x, y)$, i.e. the condition $u(0, y) = -|y|$ is not fulfilled, and therefore the solution is smooth (and satisfies the equation in classical sense) everywhere, except for the point $x = 1$, $y = 0$, which can be treated similar to the IVP case. The graph of this solution and the related characteristics are given on Fig. 3 (right). So, starting from non-smooth initial data, then "reflecting" the solution at $x = 3$, one gets certain smoothening of the solution. If we continue to "reflect" the solutions, by solving consequently the IVP and the TVP, the solution will be smooth. Fig. 4 shows the graph of the both IVP and TVP solutions.

4.3. A scalar eikonal equation

The eikonal equation, known from the geometrical optics, may arise also in shape-from-shading (in case of a vertical light source, when one has $\gamma_1 = \gamma_2 = 0$ in (4)) and computer vision problems [17]. The simple eikonal equation below, see also [5], demonstrates the difference between the IVP and the TVP solutions in the case, when the boundary surface for the both problems coincides with $\partial\Omega$. Consider the following problem with scalar x:

$$F \equiv u_x^2 - 1 = 0, \quad x \in (0, 1), \quad u(0) = 0, \quad u(1) = 0,$$

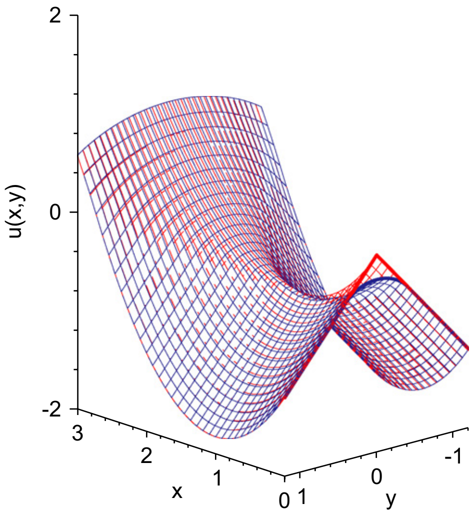

Fig. 4. Comparison of non-smooth IVP and smooth TVP solutions.

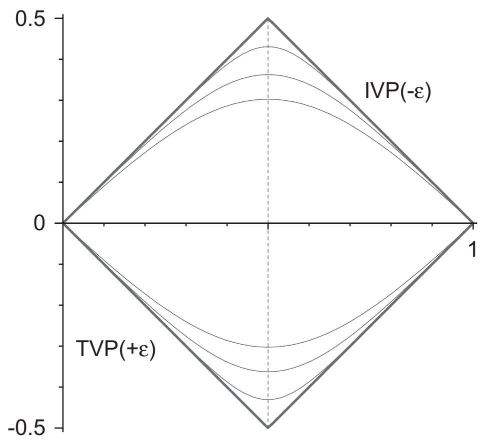

Fig. 5. IVP and TVP solutions and their approximations.

where $\Omega = (0, 1)$, and $M = \partial\Omega$ consists of two points $\{0, 1\}$. One can show that the $I(F, M)$ is the function $u = h(x)$:

$$h(x) = x, \quad x \in [0, 1/2], \qquad h(x) = 1 - x, \quad x \in [1/2, 1],$$

and the $T(F, M)$ is the function $u = -h(x)$, see Fig. 5.

For this example one can calculate the functions $\pm h(x)$ as the limits of the solutions to the following second-order ODE:

$$\pm \varepsilon u_{xx} + u_x^2 - 1 = 0, \quad u(0) = 0, \quad u(1) = 0,$$

when the positive small parameter ε tends to zero. The sign "−" generates the IVP, while the sign "+" generates the TVP. The solutions to this second order ODE for three different positive and negative values of ε are depicted on Fig. 5.

One can check the characteristic flow for the eikonal equation by writing the characteristic equation for x using $F = p^2 - 1 = 0$ $(p = u_x)$:

$$\dot{x} = F_p = 2p = 2u_x.$$

So the sign of $\dot{x}$ depends on the derivative of $u(x) = h(x)$, and the characteristic flow for the IVP goes from the set M, endpoints of the segment [0, 1], towards the inside of the interval (0, 1), while for the TVP goes towards M.

5. Conclusions

This paper discusses the initial value and the terminal value problems for non-smooth viscosity solutions of the Hamilton–Jacobi equation. If the boundary conditions are not derived from the physics of the problem, then special consideration should be given to the relevant IVP/TVP problem formulation. In some problems there is no straightforward indication from the physics of the problem, as whether IVP or TVP should be formulated, so that this issue requires further investigation.

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