Asynchronous Task Execution in Networked Control Systems Using Decentralized Event-Triggering

Xiaofeng Wang, Yu Sun, and Naira Hovakimyan

Department of Mechanical Science and Engineering
University of Illinois at Urbana-Champaign, Urbana, IL 61801

Abstract

This paper studies decentralized event-triggering in networked control systems (NCSs), where all communication/computation tasks are executed asynchronously. A decentralized event-triggering scheme is proposed under this framework. We show that if the weighted sum of all minimal task periods and all types of delays is bounded, the resulting NCS is asymptotically stable. This condition suggests a tradeoff between the system performance and the overall communication and computational resources, which can serve as basis for scheduling data traffic over the network and also computational tasks.

1. Introduction

Networked control systems (NCSs) are widely used throughout worldwide infrastructure for their advantages in terms of lower system costs due to streamlined installation procedures. In these systems, the feedback loop is closed via a communication network, and the control tasks are executed by computers. However, the introduction of digital techniques in the systems raises important issues regarding the impact of communication/computational limitations on the control system's performance. Because of the digital nature, both communication and computation tasks are executed in a discrete manner. Moreover, the hardware limitations in their turn add task delays in the loop. So one important issue in the implementation of these systems is to identify methods that more effectively use the available communication and computational resources.

Preprint submitted to System & Control Letters *October 21, 2011*

For these reasons, the timing issue in NCSs drew significant attention over the past years. Traditional approaches focus on real-time communication constraints and determine the *maximum allowable transfer interval* (MATI) between two subsequent message transmissions that ensures closed-loop stability [1, 2, 3]. However, because the MATI is computed before the system is deployed, it must ensure performance levels over all possible variations in the system. Consequently, the MATI may be conservative in the sense of being shorter than necessary. Recently, event-triggering schemes were introduced in NCSs, where tasks are executed whenever a pre-specified event occurs. Recent papers [4, 5, 6] show that event-triggering schemes can largely reduce the workload compared with the periodic task models. Prior work in decentralized implementation of event-triggering was reported in [7, 8].

A critical assumption in the prior work is that at any time instant the state information loaded in the computation of the control tasks must be consistent. In other words, once the controller receives (part of) the state information, every component in the controller must use the data synchronously. Those who receive the data earlier cannot use the data until all components are ready for using this data. Such a requirement may result in the waste of communication and computational resources. A more reasonable scenario is that every component uses the data whenever they receive it. In that case, there may be multiple versions of the state in the controller at a fixed time instant and the control inputs can be computed in an asynchronous manner. Under such a framework, this paper proposes a decentralized event-triggering scheme. This scheme allows asynchronous transmission of the state from the plant to the controller. The control inputs are computed and actuated, also in an asynchronous way. We show that under this scheme, if the weighted sum of all minimal task periods and all possible delays is bounded, the NCS is asymptotically stable.

The difference between this result and the prior work in [7] can be summarized according to the following arguments: first, we relax the requirement of consistency in the state information at the controller; second, by placing a minimal task period, our scheme theoretically ensures that the task periods are greater than a positive constant, while this property cannot be theoretically guaranteed in [7]; third, but not least, we provide a global tradeoff condition between all transmission periods and all kinds of allowable delays that ensures asymptotic stability. It indicates a relation between the system performance and the total communication and computation resources.

The prior work in [7] presents only a local tradeoff relation, i.e. the tradeoff between an individual subsystem's task periods and its associated allowable delays. To summarize, the result in [7] is a special case of this work.

The paper is organized as follows. Section 2 formulates the problem. The decentralized event-triggering scheme is introduced in Section 3. Bounds on the task periods and task delays are derived in Section 4. An illustrative example is presented in Section 5. Section 6 draws the conclusions, while all the proofs are in the appendix.

2. Problem Formulation and System Framework

2.1. Notations

We denote by $\mathbb{R}^n$ the n-dimension real vector space and by $\mathbb{R}^+$ the real positive numbers. Let $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. We also denote by $\mathbb{Z}^+$ the set of positive integers and let $\mathbb{Z}_0^+ = \mathbb{Z}^+ \cup \{0\}$. $\|\cdot\|$ denotes the Euclidean norm of a vector. For a matrix, $\|\cdot\|$ denotes the matrix norm induced by the Euclidean vector norm. The symbol $\vee$ denotes the logical operator OR, where $E_1 \vee E_2$ is true when either E_1 or E_2 is true. The symbol $\wedge$ denotes the logical operator AND, where $E_1 \wedge E_2$ is true when both E_1 and E_2 are true. The symbol $\bar{\cdot}$ denotes the logical operator NOT, where $\overline{E}$ is true when E is false. Given two functions $\alpha : \mathbb{R}^m \to \mathbb{R}^l$ and $\beta : \mathbb{R}^n \to \mathbb{R}^m$, we define $\alpha \circ \beta : \mathbb{R}^n \to \mathbb{R}^l$ to be the function $\alpha \circ \beta(x) = \alpha(\beta(x))$ with $x \in \mathbb{R}^n$. Given a set $\mathcal{A}$ with finite elements, let $|\mathcal{A}|$ be the cardinality of $\mathcal{A}$.

Given $n, m \in \mathbb{Z}^+$, let $\mathcal{N} = \{1, 2, \cdots, n\}$ and $\mathcal{M} = \{1, 2, \cdots, m\}$. For any $j \in \mathcal{M}$, $\mathcal{N}_j$ is a subset of $\mathcal{N}$. Given a collection of scalars $x_{i\to j}$ with $j \in \mathcal{M},\ i \in \mathcal{N}_j$, we define an extended vector $\langle x_{i\to j}\rangle_{j\in\mathcal{M},\ i\in\mathcal{N}_j} \in \mathbb{R}^{\sum_{j\in\mathcal{M}}|\mathcal{N}_j|}$, where

$$\langle x_{i\to j}\rangle_{j\in\mathcal{M},\ i\in\mathcal{N}_j} = \begin{pmatrix} \{x_{i\to 1}\}_{i\in\mathcal{N}_1} \\ \vdots \\ \{x_{i\to m}\}_{i\in\mathcal{N}_m} \end{pmatrix}$$

and $\{x_{i\to j}\}_{i\in\mathcal{N}_j}$ is the vector, whose entries are sorted by the index i. If the sets $\mathcal{M}$ and $\mathcal{N}_j$ are clear in context, we use $\langle x_{i\to j}\rangle$ to simplify the notation.

2.2. Problem Formulation

Consider a state-feedback NCS with n sensors and m actuators. Let $\mathcal{N} = \{1, \cdots, n\}$ and $\mathcal{M} = \{1, \cdots, m\}$. The state equation is

$$\begin{aligned} \dot{x}(t) &= F\left(x(t), u(t)\right), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x : \mathbb{R}_0^+ \to \mathbb{R}^n$ and $u : \mathbb{R}_0^+ \to \mathbb{R}^m$ are the system state and the control input, respectively, $x_0 \in \mathbb{R}^n$ is the initial condition, and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, locally Lipschitz, and $F(0, 0) = 0$.

Let $x_i : \mathbb{R}_0^+ \to \mathbb{R}$ be the i^{th} entry of the state (called the i^{th} sub-state). Then x_i satisfies

$$\dot{x}_i = F_i\left(x(t), \{u_j(t)\}_{j \in \mathcal{A}_i}\right) \tag{2}$$

where $u_j : \mathbb{R}_0^+ \to \mathbb{R}$ is the j^{th} entry of the control input (called the j^{th} sub-input) with $j \in \mathcal{M}$, $\mathcal{A}_i \in \mathcal{N}$ is the set of sub-inputs that drive x_i's dynamics, $F_i : \mathbb{R}^n \times \mathbb{R}^{|\mathcal{A}_i|} \to \mathbb{R}$ is locally Lipschitz, and

$$F\left(x(t), u(t)\right) = \begin{pmatrix} F_1\left(x(t), \{u_j(t)\}_{j \in \mathcal{A}_1}\right) \\ \vdots \\ F_n\left(x(t), \{u_j(t)\}_{j \in \mathcal{A}_n}\right) \end{pmatrix}.$$

Remark 1. *We assume x_i and u_j are scalars in this paper for notational simplicity. However, our result can be easily extended to the multi-dimensional case, where $x_i \in \mathbb{R}^{n_i}$, $u_j \in \mathbb{R}^{m_j}$ and $\sum_{i \in \mathcal{N}} n_i = n$, $\sum_{i \in \mathcal{M}} m_j = m$.*

The system structure is shown in Figure 1. We assume that sensor i ($\mathcal{S}_i$) can only continuously sample $x_i(t)$. An event generator (EG, $\mathcal{E}_i$) is located at $\mathcal{S}_i$ to determine when to transmit the sampled sub-state x_i to the controller through a real-time network. At the controller, there are m tasks; task j ($\mathcal{T}_j$) computes the j^{th} sub-input based on the received data. Once the computation is done, the new sub-inputs will be transmitted, asynchronously, back to the plant and actuated, again, through a real-time network.

We define several time instants as follows:

- $r_i^{s,k}$: the transmission release time of the k^{th} transmission generated by $\mathcal{E}_i$;

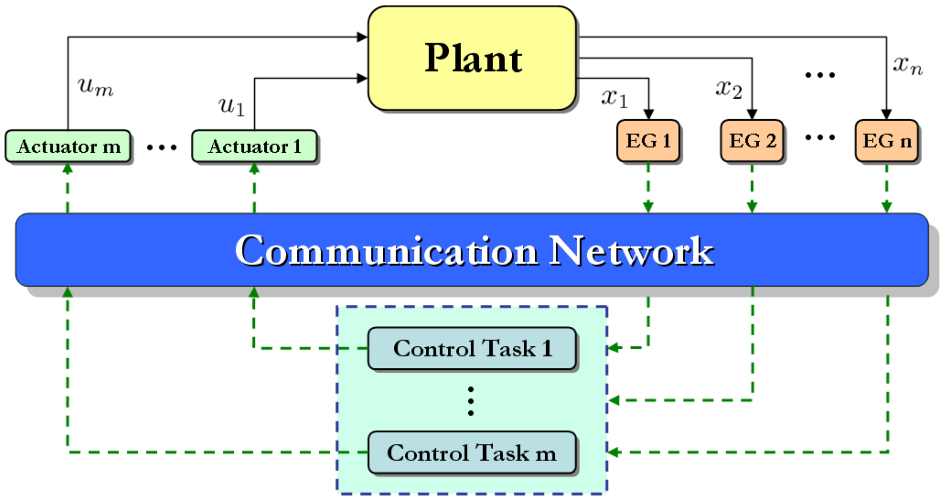

Figure 1: An event-triggered NCS

- $f_i^{s,\,k}$: the transmission finishing time of the k^{th} transmission generated by $\mathcal{E}_i$;
- $r_{i\to j}^{c,\,k}$: the release time of the computation in $\mathcal{T}_j$ that is triggered by the reception of the data in the k^{th} transmission from $\mathcal{E}_i$.
- $r_j^{c,\,l}$: the computation release time of the l^{th} computation in $\mathcal{T}_j$;
- $f_{i\to j}^{a,\,k}$: the finishing time of the actuation of $\mathcal{T}_j$'s output that is triggered by the reception of the data in the k^{th} transmission from $\mathcal{E}_i$..
- $f_j^{a,\,l}$: the finishing time of the actuation of the l^{th} output of $\mathcal{T}_j$.

Let $\mathcal{N}_j \subseteq \mathcal{N}$ be the index set of the sub-states that $\mathcal{T}_j$ receives. We know that $\{r_j^{c,\,l}\}_{l=0}^{\infty}$ and $\{f_j^{a,\,l}\}_{l=0}^{\infty}$ are the sorted sequences of $\{r_{i\to j}^{c,\,k}\}_{i\in\mathcal{N}_j,\,k\in\mathbb{Z}_0^+}$ and $\{f_{i\to j}^{a,\,k}\}_{i\in\mathcal{N}_j,\,k\in\mathbb{Z}_0^+}$, respectively.

The time-line of the task execution in the system is presented in Figure 2. At $r_i^{s,\,k}$, the data $x_i^k = x_i\left(r_i^{s,\,k}\right)$ is transmitted. The black block represents the data transmission over the network. The blue block represents the execution of the l^{th} computation in $\mathcal{T}_j$, which is triggered by the reception of x_i^k. The green block represents the transition of the output of $\mathcal{T}_j$ back to the plant. The yellow block represents the actuation of the sub-input. As a result, $r_i^{s,\,k} \le f_i^{s,\,k} \le r_{i\to j}^{c,\,k} \le f_{i\to j}^{a,\,k}$ holds for any $k \in \mathbb{Z}_0^+$.

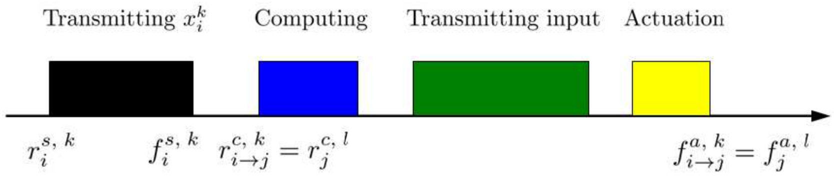

Figure 2: Time-line of tasks

This time-line shows that the impact of x_i^k through u_j appears in the plant over the time interval from $f_{i\to j}^{a,\,k}$ to $f_{i\to j}^{a,\,k+1}$. By defining $\hat{x}_{i\to j}(t) = x_i^k$ for any $t \in \left[f_{i\to j}^{a,\,k}, f_{i\to j}^{a,\,k+1}\right)$, we have

$$u_j(t) = K_j(\hat{x}_{\mathcal{N}_j\to j}(t)), \tag{3}$$

where $\hat{x}_{\mathcal{N}_j\to j}(t) = \{\hat{x}_{i\to j}(t)\}_{i\in\mathcal{N}_j}$ and $K_j : \mathbb{R}^{|\mathcal{N}_j|} \to \mathbb{R}$ is the control law with $K_j(0) = 0$. The overall control input is then

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} K_1(\hat{x}_{\mathcal{N}_1\to 1}(t)) \\ \vdots \\ K_m(\hat{x}_{\mathcal{N}_m\to m}(t)) \end{pmatrix} = K\left(\langle \hat{x}_{i\to j}(t)\rangle_{j\in\mathcal{M},\, i\in\mathcal{N}_j}\right). \tag{4}$$

The objective of this paper is to identify the real-time constraints in the communication and computation that guarantee asymptotic stability of the overall NCS.

2.3. Assumptions

Assumption 1. *There exist smooth, positive definite functions $V, \beta : \mathbb{R}^n \to \mathbb{R}_0^+$, continuous functions $\psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, and class $\mathcal{K}$ functions α_1, α_2, $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that*

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \tag{5}$$

$$\frac{\partial V(x)}{\partial x} F\left(x, K\left(\langle \hat{x}_{i\to j}\rangle_{j\in\mathcal{M},\, i\in\mathcal{N}_j}\right)\right) \le -\beta(x)\phi(\|x\|) + \beta(x)\psi\left(\|x\|, \|\langle \tilde{x}_{i\to j}\rangle_{j\in\mathcal{M},\, i\in\mathcal{N}_j}\|\right) \tag{6}$$

hold for any $x \in \mathbb{R}^n$ and $\hat{x}_{i\to j} \in \mathbb{R}$, where $\tilde{x}_{i\to j} \triangleq x_i - \hat{x}_{i\to j}$.

Remark 2. *This is a weaker assumption than input-to-state stability (ISS) with respect to* $\tilde{x}_{i\to j}$*. If* $\beta \equiv 1$ *and* ψ *only depend on* $\tilde{x}_{i\to j}$*, but are independent of* x*, inequality* (6) *will be reduced to an ISS condition from the measurement error* $\tilde{x}_{i\to j}$ *to the state* x*.*

Assumption 2. *The functions* $\phi^{-1}(s)$ *and* $\psi(a,s)$ *in Assumption 1 are locally Lipschitz with respect to* s*, i.e. given a compact set* $\Omega \subset \mathbb{R}_0^+$*, there exist* $L,\ B \in \mathbb{R}^+$ *such that the functions* $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$*,* $\psi : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ *satisfy*

$$\phi(s) \geq Ls \quad \text{and} \quad \psi(a,s) \leq Bs, \qquad \forall s, a \in \Omega. \tag{7}$$

Remark 3. *The functions* V*,* ϕ*,* ψ *and the constants* L*,* B *can be designed in a distributed way. The detailed procedure can be found in [7].*

Assumption 3. *For any* $i \in \mathcal{N}$*,* F_i *is locally Lipschitz with respect to* x *and* $\hat{x}_{i\to j}$*, i.e.*

$$\|F_i\left(x, \{K_j(\hat{x}_{\mathcal{N}_j\to j})\}_{j\in\mathcal{A}_i}\right)\| \leq a_i^\theta \|x\| + \sum_{j\in\mathcal{A}_i,\ s\in\mathcal{N}_j} b_{s\to j}^\theta \|\hat{x}_{s\to j}\| \tag{8}$$

holds for any $x \in \{x \in \mathbb{R}^n \mid \|x\| \leq \theta\}$ *and* $\hat{x}_{i\to j} \in \{\hat{x}_{i\to j} \in \mathbb{R} \mid \|\hat{x}_{i\to j}\| \leq \theta\}$*.*

3. Decentralized Event-Triggered Data Transmission

This section introduces the decentralized event-triggering scheme that triggers the data transmission. In this scheme, the i^{th} event generator, $\mathcal{E}_i$, only has the access to $x_i(t)$. The $(k+1)^{\text{st}}$ transmission by $\mathcal{E}_i$ is released when the logic rule

$$\bar{E}_1 \wedge \bar{E}_2 \vee E_3 \tag{9}$$

is true, where

$$E_1 : \|x_i(t) - x_i^k\| < \rho_i \|x_i^k\|, \tag{10}$$

$$E_2 : t - r_i^{s,\,k} < T_i^{\min}, \tag{11}$$

$$E_3 : t - r_i^{s,\,k} \geq T_i^{\max}, \tag{12}$$

and $\rho_i, T_i^{\min}, T_i^{\max} \in \mathbb{R}^+$ will be determined in the next section. Mathematically, we see that

$$r_i^{s,\,k+1} = \min_{t \geq r_i^{s,\,k}} \left\{ t \mid (\|x_i(t) - x_i^k\| \geq \rho_i \|x_i^k\|) \wedge (t - r_i^{s,\,k} \geq T_i^{\min}) \vee (t - r_i^{s,\,k} \geq T_i^{\max}) \right\}.$$

Remark 4. *The introduction of* E_1 *is to limit the measurement error* $x_i(t) - x_i^k$*. The logic rule* E_2 *is to enforce the minimal transmission period of* $\mathcal{E}_i$ *to be* $T_i^{\min}$*. In this way, continuous transmission can be avoided, which cannot be guaranteed by the scheme in [7]. The introduction of* E_3 *is for the safety of the NCS. It requires that* $\mathcal{E}_i$ *transmits at least every* $T_i^{\max}$ *unit-time, where* $T_i^{\max}$ *can be arbitrarily chosen. In general, we choose* $T_i^{\max} > T_i^{\min}$*; otherwise, if* $T_i^{\max} \leq T_i^{\min}$*, then* $\mathcal{E}_i$*'s the transmission release will always be triggered by the satisfaction of* E_3*, since* $\bar{E}_1 \wedge \bar{E}_2$ *will never be true before* E_3 *is true. In that case, the transmission becomes periodic with the period* $T_i^{\max}$*.*

Remark 5. *The rule in* (9) *means that* $\mathcal{E}_i$ *releases a transmission either when inequalities* (10) *and* (11) *are both violated (when* $\bar{E}_1 \wedge \bar{E}_2$ *is true) or it has been* $T_i^{\max}$ *unit-time since the last transmission release by* $\mathcal{E}_i$ *(when* E_3 *is true). If* $\bar{E}_1 \wedge \bar{E}_2$ *becomes true first, we have either*

$$\|x_i(t) - x_i^k\| \leq \rho_i \|x_i^k\|, \quad \forall t \in \left[r_i^{s,\,k}, r_i^{s,\,k+1}\right] \quad \textit{or} \quad r_i^{s,\,k+1} - r_i^{s,\,k} = T_i^{\min}. \tag{13}$$

If E_3 *becomes true first, then* $r_i^{s,\,k+1} - r_i^{s,\,k} = T_i^{\max}$*, which means* E_2 *is violated since* $T_i^{\max} > T_i^{\min}$*. In this case,* E_1 *must hold; otherwise* $\bar{E}_1 \wedge \bar{E}_2$ *will be true before* E_3 *is. Consequently, inequality* (10) *must hold at* $r_i^{s,\,k+1}$*. To summarize, this triggering scheme can guarantee that the statement in* (13) *is true during the runtime.*

Remark 6. *This is indeed a hybrid triggering scheme. When* E_1 *is violated too frequently (which may be because that* ρ_i *is too small or the measurement error grows too fast),* $\bar{E}_1$ *becomes true before* $\bar{E}_2$ *does. In this case, the violation of* E_2 *triggers the release, and the scheme switches to a periodic task model with the period* $T_i^{\min}$*. When it takes very long time for* E_1 *to be violated,* E_3 *becomes true before* $\bar{E}_1$ *does, which means that the satisfaction of* E_3 *triggers the release, and the scheme switches to another periodic task model with the period* $T_i^{\max}$*. Other than these two cases, if the release is triggered by the violation of* E_1*, it turns out to be a traditional sporadic event-triggered model.*

4. Real-Time Constraints on Communication and Computation

This section discusses the real-time constraints for the guarantee of asymptotic stability. We identify the parameters in the event-triggering scheme ρ_i, $T_i^{\min}$, $T_i^{\max}$, and derive bounds on the delays using a Lyapunov approach.

The analysis starts from inequality (6). Let $\tilde{x}_{i\to j}(t) = x_i(t) - \hat{x}_{i\to j}(t)$ for any $j \in \mathcal{M}$ and $i \in \mathcal{N}_j$. Inequality (6) in fact implies that for any $t \geq 0$ the following inequality

$$\begin{aligned} \dot{V} &= \left.\frac{\partial V(x)}{\partial x}\right|_{x=x(t)} F\left(x(t), K\left(\langle \hat{x}_{i\to j}(t)\rangle_{j\in\mathcal{M},\, i\in\mathcal{N}_j}\right)\right) \\ &\leq -\beta(x(t))\phi(\|x(t)\|) + \beta(x(t))\psi\left(\|x(t)\|, \|\langle\tilde{x}_{i\to j}(t)\rangle\|\right) \qquad (14) \end{aligned}$$

holds. It is easy to see from this inequality that if the communication and computation conditions are perfect ($\tilde{x}_{i\to j}(t) \equiv 0$), then $\dot{V} \leq -\beta(x(t))\phi(\|x(t)\|)$, since $\psi(s, 0) \leq 0$ holds for all $s \in \mathbb{R}_0^+$ per Assumption 2. Under imperfect conditions, we need to limit the impact of ψ on $\dot{V}$. To do this, the first step is to estimate the value of $\|\tilde{x}_{i\to j}(t)\|$, which is $\|x_i(t) - \hat{x}_{i\to j}(t)\|$. Recall that $\hat{x}_{i\to j}(t) = x_i^k$ for all $t \in \left[f_{i\to j}^{a,\,k}, f_{i\to j}^{a,\,k+1}\right)$. So, we first examine $\|x_i(t) - x_i^k\|$ over this time interval.

Lemma 1. *Consider the NCS in* (1) – (3) *with the logic rule in* (9)*, where* $\rho_i, T_i^{\max}, T_i^{\min} \in \mathbb{R}^+$ *are arbitrarily selected. Subject to Assumption 3, given* $i \in \mathcal{N}$ *and* $k \in \mathbb{Z}^+$*, if there exists* $\theta \in \mathbb{R}_0^+$ *such that* $\|x(t)\| \leq \theta$ *and* $\|\hat{x}_{i\to j}(t)\| \leq \theta$ *hold for any* $t \in \left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$*, then*

$$\|x_i(t) - x_i^k\| \leq \max\left\{\rho_i\|x_i^k\|, \theta_i T_i^{\min}\right\} + \theta_i \Delta_{i\to j} \qquad (15)$$

for any $t \in \left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$*, where*

$$\theta_i = \left(a_i^\theta + \textstyle\sum_{j\in\mathcal{A}_i,\, s\in\mathcal{N}_j} b_{s\to j}^\theta\right)\theta, \qquad (16)$$

a_i^θ*,* $b_{s\to j}^\theta$ *are Lipschitz constants of* F_i*, and* $\Delta_{i\to j} = \sup_k\left\{f_{i\to j}^{a,\,k} - r_i^{s,\,k}\right\}$.

With the bounds on $\|\tilde{x}_i(t)\|$, we are able to further bound $\dot{V}$ and show the uniform ultimate boundedness of the system.

Lemma 2. *Consider the NCS in* (1) – (3) *with the logic rule in* (9)*, where* $T_i^{\max}, T_i^{\min} \in \mathbb{R}^+$ *are arbitrarily selected. Let Assumptions 1 – 3 hold. Also assume that there exists a positive constant* θ *such that* $\|x(t)\| \leq \theta$ *and* $\|\hat{x}_{i\to j}(t)\| \leq \theta$ *hold for any* $t \geq 0$*. If* $\rho_i \in (0, 1)$ *satisfies*

$$B \max_{i\in\mathcal{N}}\left\{\frac{\rho_i\sqrt{|\mathcal{M}_i|}}{1-\rho_i}\right\} < L, \qquad (17)$$

where $L, B \in \mathbb{R}_0^+$ are defined in inequalities (7)*, and $\mathcal{M}_i \in \mathcal{M}$ is the index set of control tasks that can receive the information from $\mathcal{E}_i$, then there exists $T > 0$ such that $\|x(t)\| \leq \alpha_1^{-1} \circ \alpha_2(\sigma\theta)$ holds for all $t \geq T$, where*

$$\sigma = \frac{B\left\|\left\langle c_i(T_i^{\min}+\Delta_{i\to j})\right\rangle\right\|}{L-B\max_{i\in\mathcal{N}}\left\{\frac{\rho_i\sqrt{|\mathcal{M}_i|}}{1-\rho_i}\right\}} \quad \text{and} \quad c_i = \frac{a_i^\theta + \sum_{j\in\mathcal{A}_i,\, s\in\mathcal{N}_j} b_{s\to j}^\theta}{1-\rho_i}. \tag{18}$$

Lemma 2 means that the resulting event-triggered NCS is uniformly ultimately bounded. However, before using Lemmas 1 and 2, we need to verify the boundedness of $\|x(t)\|$ and $\|\hat{x}_{i\to j}(t)\|$. The following Lemma shows that by bounding $T_i^{\min}$ and $\Delta_{i\to j}$, the state trajectory $x(t)$ will fall into a compact set Λ defined by $\Lambda = \{x \in \mathbb{R}^n \mid V(x) \leq V(x_0)\}$, which, along with inequality (5), implies

$$\|\hat{x}_{i\to j}(t)\| \leq \sup_{t\geq 0} \|x(t)\| \leq \alpha_1^{-1} \circ V(x_0) = \theta. \tag{19}$$

Lemma 3. *Consider the NCS in* (1) *–* (3) *with the logic rule in* (9)*. Let Assumptions 1 – 3 hold. Given $\theta \in \mathbb{R}_0^+$ defined in* (19)*, if $T_i^{\max} \geq T_i^{\min}$, and $\rho_i \in (0, 1)$, $T_i^{\min} \in \mathbb{R}^+$ satisfy inequality* (17) *and the following inequality*

$$\sigma \;<\; \frac{\alpha_2^{-1} \circ V(x_0)}{\alpha_1^{-1} \circ V(x_0)}, \tag{20}$$

where $\sigma \in \mathbb{R}_0^+$ is defined in (18)*, then $x(t) \in \Lambda$ for all $t \geq 0$.*

Lemma 3 provides the condition in inequality (20) that guarantees $x(t) \in \Lambda$. It means that $\|x(t)\| \leq \theta = \alpha_1^{-1} \circ V(x_0)$ for all $t \geq 0$. Therefore, with Lemma 2, we know that the condition in (20) ensures uniform ultimate boundedness of the system with the ultimate bound $\alpha_1^{-1} \circ \alpha_2(\sigma\theta)$. It is important to note that despite the fact that the state trajectory is inside Λ, the obtained ultimate set in Lemma 3, given by $\{x \in \mathbb{R}^n \mid \|x\| \leq \alpha_1^{-1} \circ \alpha_2(\sigma\theta)\}$, is not necessarily inside Λ. To obtain asymptotic stability we need the ultimate set be inside Λ, in which case the ultimate bound will eventually shrink to the origin. Therefore, another condition, stronger than inequality (20), is needed to enforce the ultimate bound be inside Λ. This result is formally stated in the following theorem.

Theorem 1. *Assume that all the hypotheses in Lemma 3 hold except the inequality in* (20)*. Also, assume that there exist a class $\mathcal{K}$ function κ :*

$\mathbb{R}_0^+ \to \mathbb{R}_0^+$ *and a positive constant* $\mu \in (0,1)$ *such that for any* $s \in \mathbb{R}_0^+$*, the inequalities*

$$\alpha_1^{-1} \circ \alpha_2(s) \leq \mu\kappa(s) \quad \text{and} \quad \kappa(\mu s) \leq \mu\kappa(s) \tag{21}$$

hold. If the following inequality

$$\kappa\left(\sigma\alpha_1^{-1} \circ V(x_0)\right) \;<\; \alpha_2^{-1} \circ V(x_0) \tag{22}$$

holds, where σ *is defined in equation* (18)*, then the event-triggered NCS is asymptotically stable.*

Remark 7. *The introduction of* E_3 *in* (12) *ensures that the transmission of the states does not stop when the ultimate bound is achieved, i.e. when the state is inside an ultimate set. Otherwise, it is possible that* $x(t)$ *is inside the ultimate set, but* $\hat{x}_{i\to j}(t)$ *stays outside this set, in which case the proof of Theorem 1 will fail.*

Remark 8. *Theorem 1 proves asymptotic stability of the NCS. However, notice that* $\dot{V} \leq 0$ *does not have to hold for all* $t \geq 0$*. In fact, it is possible that* $\dot{V} \geq 0$ *from the moment when the state enters the ultimate set to the first time when* $\hat{x}_{i\to j}(t)$ *is inside* Λ_k*, for all* $i \in \mathcal{N}$ *and* $j \in \mathcal{A}_i$*. However, because* $x(t)$ *is bounded, the growth rate of* $V(t)$ *is bounded. Other than these intervals,* $\dot{V} \leq 0$ *always holds. Theorem 1 indicates that although* $V(t)$ *may temporarily increase, it will eventually converge to zero. This is consistent with the stability theory of hybrid systems [9].*

Remark 9. *With inequalities* (5) *and* (21)*, one can verify that inequality* (22) *implies the satisfaction of inequality* (20)*. Inequality* (22) *can be rewritten as*

$$\sigma = \frac{B\sqrt{\sum_{j\in\mathcal{M}}\sum_{i\in\mathcal{N}_i} c_i^2 (T_i^{\min} + \Delta_{i\to j})^2}}{L - B\max_{i\in\mathcal{N}}\left\{\frac{\rho_i\sqrt{|\mathcal{M}_i|}}{1-\rho_i}\right\}} < \frac{\kappa^{-1}\circ\alpha_2^{-1}\circ V(x_0)}{\alpha_1^{-1}\circ V(x_0)}. \tag{23}$$

It is obvious that for any class $\mathcal{K}$ *function* $\kappa(s)$*, we can always find positive* $T_i^{\min}$ *and* $\Delta_{i\to j}$ *ensuring inequality* (22)*.*

Remark 10. *Inequality* (22) *(or* (23)*) places a real-time constraint on communication/computation tasks. It requires the weighted sum of transmission periods and allowable delays to be bounded. One major difference of this result* *from the prior work [7] is that the work in [7] only provides a local tradeoff relation, i.e. the tradeoff between the* i^{th} *event generator* $\mathcal{E}_i$*'s transmission periods and its associated allowable delays, while this condition suggests a global tradeoff, i.e. the tradeoff between all transmission periods and all kinds of allowable delays. It indicates a relation between the system stability and the total communication and computational resources. The condition in [7] is a special case of our stability condition by assuming that* κ *is a linear function (*$\alpha_1^{-1} \circ \alpha_2$ *is locally Lipschitz) and the sampled state information is consistent in the system.*

Remark 11. *The task periods generated by this scheme will not be small because they are mostly generated by* E_1*. The minimal task period* $T_i^{\min}$ *might be small, but it is only for the purpose of providing a theoretical low bound on the task periods, which the work in [7] did not provide. By setting* $T_i^{\min} = 0$*, the scheme in [7] is recovered.*

5. An Illustrative Example

This section uses a simple example to illustrate how to apply the proposed decentralized event-triggering scheme. Consider the rotating rigid spacecraft model [10] given by

$$\begin{aligned} J_1\dot{x}_1 &= (J_2 - J_3)x_2x_3 + u_1 \\ J_2\dot{x}_2 &= (J_3 - J_1)x_3x_1 + u_2 \\ J_3\dot{x}_3 &= (J_1 - J_2)x_1x_2 + u_3, \end{aligned}$$

where x_1, x_2, x_3 are the components of the angular velocity vector along the principal axes, u_1, u_2, u_3 are the torque inputs applied about the principal axes, and J_1, J_2, J_3 are the principal moments of inertia. In the simulations, we set $J_1 = 1$, $J_2 = 2$, and $J_3 = 3$.

The controller is given by

$$\begin{aligned} u_1(t) &= -\hat{x}_{1\to1}(t) - \hat{x}_{2\to1}(t)^2 - \hat{x}_{2\to1}(t)\hat{x}_{3\to1}(t) \\ u_2(t) &= -\frac{1}{2}\hat{x}_{2\to2}(t) + \frac{1}{2}\hat{x}_{1\to2}(t)\hat{x}_{2\to2}(t) \\ u_3(t) &= -\frac{1}{3}\hat{x}_{3\to3}(t) + \frac{1}{3}\hat{x}_{1\to3}(t)\hat{x}_{2\to3}(t). \end{aligned}$$

Therefore,

$$\begin{array}{ccc} \mathcal{A}_1 = \{1\}, & \mathcal{A}_2 = \{2\}, & \mathcal{A}_3 = \{3\}, \\ \mathcal{M}_1 = \{1,2,3\}, & \mathcal{M}_2 = \{1,2,3\}, & \mathcal{M}_3 = \{1,3\}, \\ \mathcal{N}_1 = \{1,2,3\}, & \mathcal{N}_2 = \{1,2\}, & \mathcal{N}_3 = \{1,2,3\}. \end{array}$$

Consider the Lyapunov function $V(x) = x_1^2 + 2x_2^2 + 3x_3^2$. The initial condition x_0 satisfies $\|x_i(0)\| \leq 1$ for $i = 1, 2, 3$, which means $\theta = \alpha_1^{-1} \circ V(x_0) \leq \sqrt{6}$. Then, we have

$$\dot{V} \leq -2\|x\|^2 + 17.8156\|x\|\|\langle \tilde{x}_{i\to j}\rangle\|$$

and the parameters in Assumption 2 and 3 are

$$\begin{array}{llll} L = 2, & B = 17.8156, & & \\ a_1^\theta = \sqrt{6}, & b_{1\to 1}^\theta = 1, & b_{2\to 1}^\theta = \sqrt{6}, & b_{3\to 1}^\theta = \sqrt{6}, \\ a_2^\theta = \frac{\sqrt{6}}{2}, & b_{2\to 2}^\theta = \frac{1}{2}, & b_{1\to 2}^\theta = \frac{\sqrt{6}}{2}, & \\ a_3^\theta = \frac{\sqrt{6}}{3}, & b_{3\to 3}^\theta = \frac{1}{3}, & b_{1\to 3}^\theta = \frac{\sqrt{6}}{6}, & b_{2\to 3}^\theta = \frac{\sqrt{6}}{6}. \end{array}$$

To satisfy the condition in inequality (17), we choose $\rho_1 = \rho_2 = 0.05$ and $\rho_3 = 0.06$. The condition in inequality (22) is then

$$47.4808 \left\| \left\langle c_i(T_i^{\min} + \Delta_{i\to j}) \right\rangle \right\| < 1/3,$$

with $c_1 = 8.7879$, $c_2 = 3.1047$, and $c_3 = 2.0918$. To satisfy inequality (22), one possible solution is

$$\begin{array}{l} T_1^{\min} = 0.0001, \quad T_2^{\min} = 0.0002, \quad T_3^{\min} = 0.0005, \\ \Delta_{1\to j} = 0.0002, \quad \forall j \in \mathcal{M}_1, \\ \Delta_{2\to j} = 0.0003, \quad \forall j \in \mathcal{M}_2, \\ \Delta_{3\to j} = 0.0010, \quad \forall j \in \mathcal{M}_3. \end{array}$$

We run the event-triggered system for 10 seconds with $T_i^{\max} = 1$ for any $i \in \mathcal{N}$. In this simulation, we assume that the task delays are random, but bounded by $\Delta_{i\to j}$ given above. The simulation results show that the system is asymptotically stable. The transmission periods of x_1 (cross), x_2 (diamond), and x_3 (circle) are shown in Figure 3. From the plot, we can see that the periods are always greater than 0.06, which is much larger than the minimal periods. It means the transmissions are mostly triggered by E_1. It shows that the periods generated by our scheme are not conservative. Also note that the

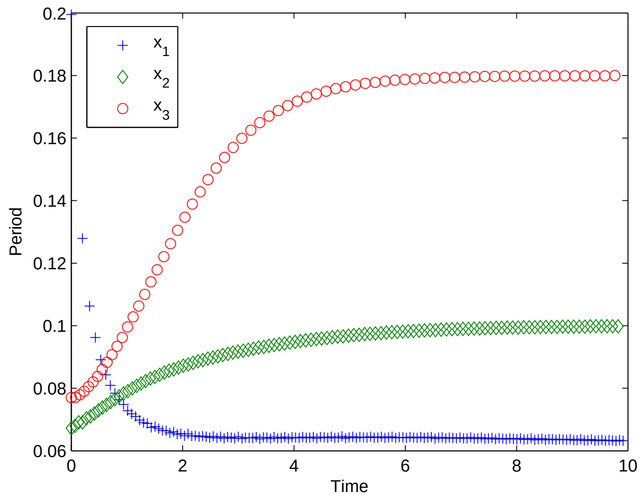

Figure 3: Transmission periods in the event-triggered NCS

periods vary in a wide range before the system approaches its equilibrium. It demonstrates the ability of event-triggering in adjusting transmission periods in response to variations in the system's states. Also notice that the periods are always greater than a positive constant. This is important because it shows that our scheme can avoid infinitely fast transmission.

6. Summary

This paper considers NCSs, where all tasks, including communication tasks and computational tasks, are executed asynchronously. A decentralized event-triggering scheme is proposed for this framework, which is guaranteed to have strictly positive transmission periods. We provide sufficient conditions under this scheme that ensure asymptotic stability of the resulting event-triggered NCS. These conditions suggest a tradeoff among all minimal task periods and bounds on all types of allowable delays.

There are several open problems in this framework, such as considering quantization effects, output-feedback schemes, to name a few. One important problem is how to take advantage of the conditions in inequalities (17) and (22) to schedule the data transmissions and computation. One possible way is to consider earliest deadline first (EDF) algorithm [11]. The entire schedulability problem can be treated as two coupled sub-problems: one is the schedulability of the data transmission and the other one is the schedulability of the computation tasks. The periods of computation tasks depend on the communication protocols. The minimal transmission period $T_i^{\min}$ can be used for the schedulability analysis. Upon obtaining the schedulability condition, we can combine it with the performance condition obtained in this paper and optimize the utilization of the communication and computational resources.

References

[1] G. Walsh, H. Ye, and L. Bushnell, "Stability analysis of networked control systems," *IEEE Transactions on Control Systems Technology*, vol. 10, no. 3, pp. 438–446, 2002.

[2] D. Nesic and A. Teel, "Input-output stability properties of networked control systems," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1650–1667, 2004.

[3] W. Heemels, A. Teel, N. van de Wouw, and D. Nesic, "Networked control systems with communication constraints: tradeoffs between sampling intervals, delays and performance," to appear in *IEEE Transactions on Automatic Control*.

[4] P. Tabuada, "Event-Triggered Real-Time Scheduling of Stabilizing Control Tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, 2007.

[5] X. Wang and M. Lemmon, "Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability," *IEEE Transactions on Automatic Control*.

[6] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042.

[7] X. Wang and M. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586–601, 2011.

[8] M. Mazo and P. Tabuada, "On event-triggered and self-triggered control over sensor/actuator networks," in *Proceedings of the 47th Conference on Decision and Control*, 2008.

[9] M. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[10] H. Khalil, *Nonlinear systems.* Prentice Hall Upper Saddle River, NJ, 2002.

[11] C. Liu and J. Layland, "Scheduling for multiprogramming in a hard-real-time environment," *Journal of the Association for Computing Machinery*, vol. 20, no. 1, pp. 46–61, 1973.

Appendix A. Proof of Lemma 1

Proof. Since $\|x(t)\|$ and $\|\hat{x}_{i\to j}(t)\|$ are bounded over $\left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$, based on equation (2), we have $\|\dot{x}_i(t)\| = \|F_i\left(x(t), \{K_j(\hat{x}_{\mathcal{N}_j\to j}(t))\}_{j\in\mathcal{A}_i}\right)\|$. By Assumption 3, F_i is locally Lipschitz. So we further re-write the preceding inequality as

$$\begin{aligned}\|\dot{x}_i(t)\| &\le a_i^\theta\|x(t)\| + \sum_{j\in\mathcal{A}_i,\, s\in\mathcal{N}_j} b_{s\to j}^\theta\|\hat{x}_{s\to j}(t)\| \\ &\le \left(a_i^\theta + \sum_{j\in\mathcal{A}_i,\, s\in\mathcal{N}_j} b_{s\to j}^\theta\right)\theta = \theta_i,\end{aligned}$$

where the last inequality is obtained using $\|\hat{x}_{i\to j}(t)\| \le \|x(t)\| \le \theta$. So $\frac{d}{dt}\|x_i(t) - x_i^k\| \le \theta_i$ holds for any $t \in \left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$. Solving this inequality for any $t \in \left[r_i^{s,\,k}, r_i^{s,\,k+1}\right]$, we obtain $\|x_i(t) - x_i^k\| \le \theta_i\left(t - r_i^{s,\,k}\right)$.

By the event-triggering scheme with the discussion in remark 5, we know that the condition in (13) holds. Combining this with the preceding inequality implies

$$\|x_i(t) - x_i^k\| \le \max\left\{\rho_i\|x_i^k\|, \theta_i T_i^{\min}\right\}, \quad \forall t \in \left[r_i^{s,\,k}, r_i^{s,\,k+1}\right]. \tag{A.1}$$

Again, solving $\frac{d}{dt}\|x_i(t) - x_i^k\| \le \theta_i$ for any $t \in \left[r_i^{s,\,k+1}, f_{i\to j}^{a,\,k+1}\right)$, we obtain

$$\begin{aligned}\|x_i(t) - x_i^k\| &\le \|x_i^{k+1} - x_i^k\| + \theta_i\left(t - r_i^{s,\,k+1}\right) \\ &\le \max\left\{\rho_i\|x_i^k\|, \theta_i T_i^{\min}\right\} + \theta_i\Delta_{i\to j}, \end{aligned} \tag{A.2}$$

where we use inequality (A.1). Combining inequalities (A.1) and (A.2) completes the proof. □

Appendix B. Proof of Lemma 2

Proof. Since $\|x(t)\| \leq \theta$ and $\|\hat{x}_{i\to j}(t)\| \leq \theta$ for any $t \geq 0$, we apply Lemma 1 to obtain

$$\begin{aligned} \|x_i(t) - x_i^k\| &\leq \max\left\{\rho_i\|x_i^k\|, \theta_i T_i^{\min}\right\} + \theta_i \Delta_{i\to j} \\ &\leq \rho_i\|x_i^k\| + \theta_i T_i^{\min} + \theta_i \Delta_{i\to j}, \end{aligned}$$

which holds for all $t \in \left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$, where θ_i is defined in equation (16). Then we have

$$\begin{aligned} (1-\rho_i)\|x_i(t) - x_i^k\| &\leq \rho_i\|x_i^k\| + \theta_i T_i^{\min} + \theta_i \Delta_{i\to j} - \rho_i\|x_i(t) - x_i^k\| \\ &\leq \theta_i(T_i^{\min} + \Delta_{i\to j}) + \rho_i\|x_i(t)\|, \end{aligned} \tag{B.1}$$

which holds for all $t \in \left[r_i^{s,\,k}, f_{i\to j}^{a,\,k+1}\right)$, and therefore for all $t \in \left[f_{i\to j}^{a,\,k}, f_{i\to j}^{a,\,k+1}\right)$.

We now consider $\dot{V}$ at time t. For $t \geq 0$, any $j \in \mathcal{M}$, and any $i \in \mathcal{N}_j$, there must exist $k_{i\to j} \in \mathbb{Z}^+$ such that $t \in \left[f_{i\to j}^{a,\,k_{i\to j}}, f_{i\to j}^{a,\,k_{i\to j}+1}\right)$ holds. Therefore, at this specific time instant t, the control input in $\mathcal{T}_j$ is computed based on $\left\{x_i^{k_{i\to j}}\right\}_{i\in\mathcal{N}_j}$. By equation (6), the time derivative of V at time t satisfies

$$\begin{aligned} \dot{V} &\leq -\beta(x(t))\left(\phi(\|x(t)\|) - \psi\left(\|x(t)\|, \|\langle x_i(t) - \hat{x}_{i\to j}(t)\rangle\|\right)\right) \\ &= -\beta(x(t))\left(\phi(\|x(t)\|) - \psi\left(\|x(t)\|, \left\|\left\langle x_i(t) - x_i^{k_{i\to j}}\right\rangle\right\|\right)\right) \\ &\leq -\beta(x(t))\left(\phi(\|x(t)\|) - \psi\left(\|x(t)\|, \left\|\left\langle \frac{\theta_i(T_i^{\min} + \Delta_{i\to j}) + \rho_i\|x_i(t)\|}{1-\rho_i}\right\rangle\right\|\right)\right), \end{aligned}$$

where the last inequality is obtained using inequality (B.1).

Since $\|x(t)\| \leq \theta$ for any $t \geq 0$, we can apply Assumption 2 to the preceding inequality and obtain

$$\begin{aligned} \dot{V} &\leq -\beta(x(t))\left(L\|x(t)\| - B\left\|\left\langle \frac{\theta_i(T_i^{\min} + \Delta_{i\to j})}{1-\rho_i}\right\rangle\right\| - B\left\|\left\langle \frac{\rho_i\|x_i(t)\|}{1-\rho_i}\right\rangle\right\|\right) \\ &\leq -\beta(x(t))\left(L\|x(t)\| - d - Bp\|x(t)\|\right) \end{aligned} \tag{B.2}$$

where $p = \max_{i\in\mathcal{N}}\left\{\frac{\rho_i\sqrt{|\mathcal{M}_i|}}{1-\rho_i}\right\}$, $d = B\left\|\left\langle\frac{\theta_i(T_i^{\min}+\Delta_{i\to j})}{1-\rho_i}\right\rangle\right\|$, and the second inequality is obtained using the fact $\left\|\left\langle\frac{\rho_i\|x_i(t)\|}{1-\rho_i}\right\rangle\right\| \leq p\|x(t)\|$. Therefore there must exist $T \geq 0$ such that $\|x(t)\| \leq \alpha_1^{-1}\circ\alpha_2\left(\frac{d}{L-Bp}\right)$ holds for all $t \geq T$, as shown in [10, pp. 169]. With the definition of θ_i in equation (16), we know $\frac{d}{L-Bp} = \sigma\theta$, which implies $\|x(t)\| \leq \alpha_1^{-1}\circ\alpha_2(\sigma\theta)$ for all $t \geq T$. □

Appendix C. Proof of Lemma 3

Proof. First, we prove that $V(x(t)) \leq V(x_0)$ holds for all $t > 0$ by contradiction. Suppose that there is a time instant $\bar{t} > 0$, such that $V(x(\bar{t})) > V(x_0)$. Notice that before the first time when the inequality (10) is violated for any $i \in \mathcal{N}$, the inequality $\dot{V} < 0$ holds. Therefore, there must exist a time instant $t^* \in (0, \bar{t})$ such that

$$V(x(t)) < V(x(t^*)) = V(x_0), \quad \forall t \in [0, t^*) \tag{C.1}$$
$$\dot{V} > 0, \quad \forall t \in (t^* - \epsilon, t^*], \tag{C.2}$$

where ϵ is a small positive constant. These inequalities imply

$$x(t) \in \Lambda \quad \text{and} \quad \|x_i(t)\| \leq \|x(t)\| \leq \alpha_1^{-1}\circ V(x_0) = \theta \tag{C.3}$$

for all $t \in [0, t^*]$. Following a similar analysis as in the proof of Lemma 2, we obtain

$$\dot{V} \leq -\beta(x(t))\left(L\|x(t)\| - d - Bp\|x(t)\|\right) \tag{C.4}$$

for all $t \in [0, t^*]$, where $p = \max_{i\in\mathcal{N}}\left\{\frac{\rho_i\sqrt{|\mathcal{M}_i|}}{1-\rho_i}\right\}$ and

$$d = B\left\|\left\langle\frac{\theta_i(T_i^{\min} + \Delta_{i\to j})}{1-\rho_i}\right\rangle\right\|.$$

By inequality (C.2), we know $\dot{V} > 0$ for any $t \in (t^* - \epsilon, t^*]$. Combining this with the preceding inequality yields

$$0 < \dot{V} \leq -\beta(x(t))\left(L\|x(t)\| - d - Bp\|x(t)\|\right)$$

for any $t \in (t^* - \epsilon, t^*]$, which implies

$$\|x(t)\| < \frac{d}{L - Bp}$$

for any $t \in (t^* - \epsilon, t^*]$. Applying the definition of θ_i in (16) into d in the preceding inequality yields

$$\begin{aligned} \|x(t)\| &< \frac{\alpha_1^{-1} \circ V(x_0) B \left\| \left\langle \frac{a_i^\theta + \sum_{j \in \mathcal{A}_i,\, s \in \mathcal{N}_j} b_{s \to j}^\theta}{1-\rho_i} (T_i^{\min} + \Delta_{i \to j}) \right\rangle \right\|}{L - Bp} \\ &= \sigma \alpha_1^{-1} \circ V(x_0) \end{aligned}$$

for any $t \in (t^* - \epsilon, t^*]$, where σ is defined in equation (18). By inequality (20), we have

$$\frac{\sigma \alpha_1^{-1} \circ V(x_0)}{\alpha_2^{-1} \circ V(x_0)} < 1.$$

Then we have $\|x(t^*)\| < \alpha_2^{-1} \circ V(x_0)$, and therefore $V(x(t^*)) \leq \alpha_2(\|x(t^*)\|) < V(x_0)$, which contradicts the inequality (C.1). Therefore, $V(x(t)) \leq V(x_0)$ always holds. □

Appendix D. Proof of Theorem 1

Proof. By inequalities (21) and (22), we know

$$\begin{aligned} \sigma \alpha_1^{-1} \circ V(x_0) &\leq \alpha_1^{-1} \circ \alpha_2 \left(\sigma \alpha_1^{-1} \circ V(x_0) \right) \\ &\leq \mu \kappa \left(\sigma \alpha_1^{-1} \circ V(x_0) \right) \\ &< \kappa \left(\sigma \alpha_1^{-1} \circ V(x_0) \right) < \alpha_2^{-1} \circ V(x_0). \end{aligned}$$

Then by Lemma 3, we know $x(t) \in \Lambda$ for all $t \geq 0$, which means

$$\|x(t)\| \leq \alpha_1^{-1} \circ V(x_0),$$

since $\alpha_1(\|x\|) \leq V(x)$ and $\|\hat{x}_{i \to j}(t)\| \leq \sup_{t \geq 0} \|x(t)\| \leq \alpha_1^{-1} \circ V(x_0)$. So we can apply Lemma 2 to show that there exists $t_1 > 0$ such that

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1} \circ \alpha_2 \left(\sigma \alpha_1^{-1} \circ V(x_0) \right) \\ &\leq \mu \kappa \left(\sigma \alpha_1^{-1} \circ V(x_0) \right) \\ &\leq \mu \alpha_2^{-1} \circ V(x_0) \leq \mu \alpha_1^{-1} \circ V(x_0) \end{aligned}$$

holds for any $t \geq t_1$, where the third inequality is obtained using inequality (22). Since $\mathcal{S}_i$ transmits at least every $T_i^{\max}$ unit-time according to E_3, we know that there exists $s_1 \geq t_1$ such that

$$\|\hat{x}_{i\to j}(t)\| \leq \mu\alpha_1^{-1} \circ V(x_0)$$

holds for all $t \geq s_1$. Then we can re-apply Lemma 2 to get the new ultimate bound on $\|x(t)\|$, i.e. there exists $t_2 > s_1$ such that

$$\begin{aligned} \|x(t)\| &\leq \alpha_1^{-1} \circ \alpha_2 \left(\mu\alpha_1^{-1} \circ V(x_0)\right) \\ &\leq \mu\kappa \left(\mu\alpha_1^{-1} \circ V(x_0)\right) \\ &\leq \mu^2\kappa \left(\alpha_1^{-1} \circ V(x_0)\right) \\ &\leq \mu^2\alpha_2^{-1} \circ V(x_0) \quad \leq \quad \mu^2\alpha_1^{-1} \circ V(x_0) \end{aligned}$$

holds for all $t \geq t_2$. Also, we know that there exists $s_2 \geq t_2$ such that $\|\hat{x}_{i\to j}(t)\| \leq \mu^2\alpha_1^{-1} \circ V(x_0)$ for any $t \geq s_2$.

Keeping this procedure, we know that there exists $s_k > 0$, such that

$$\|x(t)\| \leq \mu^k\alpha_1^{-1} \circ V(x_0)$$

holds for all $t \geq s_k$. Since $\mu \in (0, 1)$, as $k \to \infty$, the preceding inequality implies $x(t) \to 0$, which implies asymptotic stability of the NCS. □